EFFECT OF NON-LINEAR LOWER ORDER TERMS IN QUASILINEAR EQUATIONS INVOLVING THE $p(\cdot)$-LAPLACIAN

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Abstract. In this work, we study the existence of $W_0^{1,p(\cdot)}$-solutions to the following boundary value problem involving the $p(\cdot)$-Laplacian operator:

$$\begin{cases} -\Delta_{p(x)}u + |\nabla u|^{q(x)} = g(x)u^{\eta(x)} + f(x), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

under appropriate ranges on the variable exponents. We give assumptions on f and g in terms of the growth exponents q and η under which the above problem has a solution in $W_0^{1,p(\cdot)}$.

1. Introduction

The contribution of the article is to give conditions on the data f and g to guarantee the existence of weak solutions in the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ to boundary value problems with the $p(\cdot)$-Laplacian operator:

$$\begin{cases} -\Delta_{p(x)}u + |\nabla u|^{q(x)} = g(x)u^{\eta(x)} + f(x), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Our results extend the analysis of [2] and [24] to the non-standard framework with the difference that we look for solutions in $W_0^{1,p(\cdot)}(\Omega)$ and not only in $W_0^{1,q(\cdot)}(\Omega)$. However, to obtain the desired results we should impose somewhat more regularity on the data.

We always assume that $\Omega \subset \mathbb{R}^N$ is open, bounded and connected with smooth boundary, and that the exponents satisfy:

$$p, q, \eta \in \mathcal{C}(\overline{\Omega}),\ p^- := \min_{\Omega} p(\cdot) > 1,\ p^+ := \max_{\Omega} p(\cdot) < N,\ q^- > \eta^+,\ 1 \leq \eta(\cdot) < q^*(\cdot) - 1. \tag{1.2}$$

The main result of the paper Theorem 3.2 states the existence of solutions to (1.1) under the following additional assumptions on the exponents:

$$\max\{p(\cdot) - 1, 1\} \leq q(\cdot) < p(\cdot), \tag{1.3}$$

and appropriate integrability conditions on f and g. Moreover, in Theorem 3.4 we also state existence of $W_0^{1,p(\cdot)}$-solutions for the degenerate case $p(\cdot) \geq 2$ with natural

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growth in the gradient:

$$\begin{cases} -\Delta_{p(x)}u + |\nabla u|^{p(x)} = g(x)u^{\eta(x)} + f(x), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

under slighter conditions on the non-negative data f and g. Indeed, in this case we just require $f \in L^1(\Omega)$ and $g \in L^{(q^*(\cdot)/\eta(\cdot))'}(\Omega)$, recovering results in the cases of the Laplacian [2] and of the p-Laplacian [24]. The existence of solutions in $W_0^{1,p(\cdot)}$ to (1.1) and (1.4) does not follow from the general results from [1] and [22], which are based on Leray-Lions' Theorem or Brezis' Theorem for pseudo-monotone operators in separable reflexive spaces. Here, we are not able to use that technique due to the higher range of the exponents (coerciveness is not obtained in general). Thus, our approach is different, uses truncations and hence is closer to the arguments in [2] and [7] (see also [8], [9], [10],[25] and the reference therein). However, limitations derived from the theory of equations with non-standard growth force to introduce variations in the proof of the main results.

Recent systematic study of partial differential equations with variable exponents was motivated by the description of models in electrorheological and thermorheological fluids, image processing, or robotics. As an illustrative example, we discuss the model [12] for image restoration. Let us consider an input I that corresponds to shades of gray in a domain $\Omega \subset \mathbb{R}^2$. We assume that I is made up of the true image u corrupted by the noise and that the noise is additive. Thus, the effect of the noise can be eliminated by smoothing the input, which corresponds to minimizing the energy:

$$E_1(u) = \int_\Omega |\nabla u(x)|^2 + |u(x) - I(x)|^2 dx.$$

Unfortunately, smoothing destroys the small details of the image, so this procedure is not useful. A better approach is the total variation smoothing. Since an edge in the image gives rise to a very large gradient, the level sets around the edge are very distinct, so this method does a good job of preserving edges. Total variation smoothing corresponds to minimizing the energy:

$$E_2(u) = \int_\Omega |\nabla u(x)| + |u(x) - I(x)|^2 dx.$$

However, total variation smoothing not only preserves edges, but it also creates edges where there were none in the original image. The suggestion of [12] was to ensure total variation smoothing ($p = 1$) along edges and Gaussian smoothing ($p = 2$) in homogeneous regions. Furthermore, it employs anisotropic diffusion ($1 < p < 2$) in regions which may be piecewise smooth or in which the difference between noise and edges is difficult to distinguish. Specifically, they proposed to minimize:

$$E(u) = \int_\Omega \phi(x, \nabla u) + (u - I)^2 dx$$

where:

$$\phi(x,\xi) := \begin{cases} \frac{1}{p(x)}|\xi|^{p(x)}, & \text{if } |\xi| \leq \beta, \\ |\xi| - C(\beta, p(x)), & \text{if } |\xi| > \beta \end{cases}$$

where $\beta > 0$ and $1 \leq p(x) \leq 2$. According to [12], the main benefit of this model is the manner in which it accommodates the local image information. Where the gradient is sufficiently large (i.e. likely edges), only total variation based diffusion will be used. Where the gradient is close to zero (i.e. homogeneous regions), the model is isotropic. At all other locations, the filtering is somewhere between Gaussian and total variation based. When minimizing over u of bounded variations, satisfying given Dirichlet conditions, the associated flow is:

$$u_t - \operatorname{div}(\phi_r(x, \nabla u)) + 2(u - I) = 0, \text{ in } \Omega \times [0, T],$$

with $u(x, 0) = I(x)$, u satisfying the prescribed boundary conditions. Hence, the above model is directly related to the study of PDE's with the $p(\cdot)$-Laplacian operator:

$$\Delta_{p(x)} u := \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right).$$

Classical references for existence and regularity of solution for $p(\cdot)$-Laplacian Dirichlet problems are [21], [20] and [19], among others. Existence and uniqueness for $p(x)$-laplace equations with zero order terms have been given in [17]. Some new criteria to guarantee the existence of multiple solutions were recently given in [26].

Elliptic equations with first order terms have been largely studied in the literature. It has been shown in [6] that the equation:

$$-\Delta u = \lambda \frac{u}{|x|^2} + f(x), \qquad \text{in a bounded } \Omega,\ 0 \in \Omega,$$

has in general no solution for a positive $f \in L^1(\Omega)$. Indeed, in [4, Theorem 2.3], it is proved that a sufficient and necessary condition for existence (for $f \in L^1(\Omega)$) is that:

$$|x|^{-2} f \in L^1(\Omega).$$

In contrast, by adding a quadratic gradient term on the left-hand side, solutions do exist for any $\lambda > 0$ and non-negative $f \in L^1(\Omega)$ (see [3]). This phenomenon has been studied in depth in the reference [2] for problems of the form:

$$\left\{ \begin{array}{rl} -\Delta u + |\nabla u|^q = \lambda g(x) u + f(x), & \text{in } \Omega, \\ u > 0, & \text{on } \Omega. \\ u = 0, & \text{on } \partial\Omega. \end{array} \right. \tag{1.5}$$

for the range $1 \leq q \leq 2$. Indeed, it is proved that, for $q \in (1, 2]$, and if $g \in L^1(\Omega)$ satisfies:

$$g \geq 0,\ g \neq 0, \text{ and } C(g, q) := \inf_{\phi \in W_0^{1,q}(\Omega)} \frac{\left(\int_\Omega |\nabla \phi|^q dx\right)^{1/q}}{\int_\Omega g |\phi| dx} > 0,$$

then Problem (1.5) admits a distributional solution in $W_0^{1,q}(\Omega)$ for any non-negative $f \in L^1(\Omega)$, and any $\lambda \geq 0$. Under higher integrability assumptions on f and g, it is possible to get solutions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ (see [2, Theorem 2.4]). The case of a convex function of the gradient $\varphi(\nabla u)$ ($q \geq 2$ in (1.5)), f Lipschitz and $\lambda = 0$ has been treated in [23]. Regarding equations with the p-Laplacian operator, we refer the reader to [24].

The paper is organized as follows. In section 2 we collect some preliminaries results in the framework of variable exponent spaces. In section 3 we introduce the main results of the paper. In section 4 we prove Theorem 3.2 and, finally, in section 5 we give the proof of Theorem 3.4.

2. Preliminaries

In this section we introduce basic definitions and preliminary results related to spaces of variable exponent and the related theory of differential equations.

Let:

$$\mathcal{C}_+(\overline{\Omega}) := \left\{p \in \mathcal{C}(\overline{\Omega}) : p(x) > 1 \ \text{ for any } \ x \in \overline{\Omega}\right\}$$
$$p^- := \min_{\overline{\Omega}} p(\cdot), \quad p^+ := \max_{\overline{\Omega}} p(\cdot).$$

We always assume that the variable exponents p are taking in $\mathcal{C}_+(\overline{\Omega})$ and satisfy that there is $C > 0$ so that:

$$|p(x) - p(y)| \leq C \frac{1}{|\log|x-y||}, \quad \text{for all } x, y \in \Omega \quad x \neq y. \tag{2.1}$$

We also define the variable exponent Lebesgue space by:

$$L^{p(\cdot)}(\Omega) := \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

A norm in $L^{p(\cdot)}(\Omega)$ is defined as follows:

$$\|u\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where:

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

For the next results see [15].

Theorem 2.1 (Hölder's inequality)**.** *The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a separable, uniform convex Banach space. For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ there holds:*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Proposition 2.2. *Let:*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx, \quad u \in L^{p(\cdot)}(\Omega)$$

be the convex modular. Then the following assertions hold:

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ *(resp. $= 1, > 1$) if and only if $\rho(u) < 1$ (resp. $= 1, > 1$);*

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ *implies* $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$, *and* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ *implies* $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$*;*

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} \to 0$ *if and only if* $\rho(u) \to 0$*, and* $\|u\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *if and only if* $\rho(u) \to \infty$*.*

We now give a useful result in order to work with different variable Lebesgue exponents (see [16]).

Lemma 2.3. *Suppose that* $p, q \in \mathcal{C}_+(\overline{\Omega})$ *and that* $1 \le p(\cdot)q(\cdot) \le +\infty$ *for all* $x \in \Omega$. *Let* $f \in L^{p(\cdot)q(\cdot)}(\Omega)$, f *not identically* 0. *Then:*

(i) $\|f\|^{p^+}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|f^{p(\cdot)}\|_{L^{q(\cdot)}(\Omega)} \le \|f\|^{p^-}_{L^{p(\cdot)q(\cdot)}(\Omega)}$ *if* $\|f\|_{L^{p(\cdot)q(\cdot)}(\Omega)} \le 1$;

(ii) $\|f\|^{p^-}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|f^{p(\cdot)}\|_{L^{q(\cdot)}(\Omega)} \le \|f\|^{p^+}_{L^{p(\cdot)q(\cdot)}(\Omega)}$ *if* $\|f\|_{L^{p(\cdot)q(\cdot)}(\Omega)} \ge 1$.

The Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as follows (∇u denotes the distributional gradient):

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

equipped with the norm:

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The following Sobolev Embedding Theorem for variable exponent spaces holds.

Theorem 2.4. *If* $p^+ < N$, *then*

$$0 < S(p(\cdot), q(\cdot), \Omega) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{L^{p(\cdot)}(\Omega)}}{\|v\|_{L^{q(\cdot)}(\Omega)}},$$

for all

$$1 \le q(\cdot) \le p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)}.$$

Remark 2.5. We need the $q(\cdot)$ exponent to be uniformly subcritical, i.e. $\inf_\Omega (p^*(\cdot) - q(\cdot)) > 0$ to assure that $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is still compact.

We recall that the $p(\cdot)$-Laplace operator is given by:

$$-\Delta_{p(x)} u := -\text{div } \left(|\nabla u|^{p(x)-2} \nabla u \right).$$

Let $X = W_0^{1,p(\cdot)}(\Omega)$. The operator $-\Delta_{p(x)}$ is the weak derivative of the functional $J : X \to \mathbb{R}$:

$$J(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx$$

in the sense that if $L = J' : X \to X^*$ then:

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad u, v \in X.$$

We want to mention that the sharp regularity for the gradient in the context of variable exponent spaces was obtained in [5, 14]. We also recall the following properties.

Theorem 2.6. *Let* $X = W_0^{1,p(\cdot)}(\Omega)$. *Then:*

(i) $L : X \to X^*$ *is continuous, bounded and strictly monotone;*

(ii) *L is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in X and:*

$$\limsup_{n\to\infty}(L(u_n)-L(u),u_n-u)\leq 0$$

then $u_n \to u$ in X;

(iii) *L is a homeomorphism.*

We also quote the following useful lemma [1, Lemma 3.3].

Lemma 2.7. *Let $1<r(\cdot)<\infty$, $g\in L^{r(\cdot)}(\Omega)$ and $g_n\in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{L^{r(\cdot)}(\Omega)}\leq C$. If $g_n(x)\to g(x)$ a.e. in Ω, then $g_n\rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

The next generalization of Lemma 1.17 in [13] to the variable exponent setting holds true.

Lemma 2.8. *Suppose $p(\cdot)\in(1,+\infty)$. Let $\{u_\epsilon\}_\epsilon$ be a weakly convergent sequence in $L^{p(\cdot)}(\Omega)$ with limit u and let $\{\phi_\epsilon\}_\epsilon$ be a bounded sequence in $L^\infty(\Omega)$ with limit ϕ a.e in Ω. Then $u_\epsilon\phi_\epsilon\rightharpoonup u\phi$ weakly in $L^{p(\cdot)}(\Omega)$.*

3. Main results

We now give the main results of the paper which state the existence of solutions to Problems (1.1) and (1.4). We start giving the notion of solution that we shall employ in the sequel.

Definition 3.1. *We say that $u\in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to Problem (1.1) or (1.4) if $gu^{\eta(\cdot)}\in L^1_{loc}(\Omega)$ and:*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\phi\,dx+\int_\Omega|\nabla u|^{q(x)}\phi\,dx=\int_\Omega g(x)u^{\eta(x)}\phi\,dx+\int_\Omega f(x)\phi\,dx$$

for all $\phi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

The main contribution of the article is the following existence result for the Dirichlet problem (1.1).

Theorem 3.2. *Assume (1.2) and (1.3). Let $f\in L^{q_0}(\Omega)$ be non-negative and $g\in L^{q_1(\cdot)}(\Omega)$, $g\gneqq 0$, where:*

$$q_0:=\left(\frac{Nq^-}{N-q^-}\right)',\qquad q_1(\cdot):=\left(\frac{q^*(\cdot)}{\eta(\cdot)+1}\right)'. \tag{3.1}$$

Then there is a weak solution $u\in W_0^{1,p(\cdot)}(\Omega)$ to (1.1).

Remark 3.3. Observe that if $g\in L^{q_1(\cdot)}(\Omega)$ then $g\in L^{\left(\frac{q^*(\cdot)}{\eta(\cdot)}\right)'}(\Omega)$. So, for any $\phi\in W_0^{1,q(\cdot)}(\Omega)$, we derive $\phi\in L^{q^*(\cdot)}(\Omega)$ and hence $\phi^{\eta(\cdot)}\in L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$. By the assumption on g we obtain:

$$\begin{aligned}\|g^{\frac{1}{\eta(\cdot)}}\phi\|_{L^{\eta(\cdot)}(\Omega)}&\leq C_0(\eta,q)\|g^{\frac{1}{\eta(\cdot)}}\|_{L^{(q^*(\cdot)/\eta(\cdot))'\eta}(\Omega)}\|\phi\|_{L^{q^*(\cdot)}(\Omega)}\\&=C_0(g,\eta,q)\|\phi\|_{L^{q^*(\cdot)}(\Omega)}\\&\leq C_0(g,\eta,q)\|\nabla\phi\|_{L^{q(\cdot)}(\Omega)},\end{aligned}$$

where we have used Lemma 2.3. As a result:

$$C(g,\eta,q) := \inf_{\phi\in W_0^{1,q(\cdot)}(\Omega)} \frac{\|\nabla\phi\|_{L^{q(\cdot)}(\Omega)}}{\|g^{\frac{1}{\eta(\cdot)}}\phi\|_{L^{\eta(\cdot)}(\Omega)}} > 0. \tag{3.2}$$

For the case $p(x) = q(x)$ for all $x \in \Omega$ we have the next result. Regarding the assumption $p(\cdot) \geq 2$, we refer the reader to Remark 5.1.

Theorem 3.4. *Assume* (1.2) *and* $p(\cdot) \geq 2$. *Let* $f \in L^1(\Omega)$ *be non-negative and* $g \in L^{\left(\frac{q^*(\cdot)}{\eta(\cdot)}\right)'}(\Omega)$, $g \gneqq 0$. *Then there is a weak solution* $u \in W_0^{1,p(\cdot)}(\Omega)$ *to* (1.4).

The constant case is a straightforward consequence of the above results (compare to [24]).

Corollary 3.5. *Assume* $1 < p < N$ *and:*

$$\max\left\{1, p-1, \frac{Np}{N+p}\right\} < q < p. \tag{3.3}$$

For non-negative $f \in L^{(q^*)'}(\Omega)$ *and* $g \in L^{(q^*/p)'}(\Omega)$, $g \gneqq 0$, *there is a non-negative solution* $u \in W_0^{1,p}(\Omega)$ *of:*

$$\begin{cases} -\Delta_p u + |\nabla u|^q = g(x)u^{p-1} + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Notice that condition $q \geq Np/(N+p)$ in (3.3) is needed in order to have (1.2) for $\eta = p - 1$.

Corollary 3.6. *Assume* $q = p$ *and* $2 \leq p < N$. *Let* $f \in L^1(\Omega)$ *and* $g \in L^{q^*/(p-1)}(\Omega)$ *be non-negative,* $g \gneqq 0$. *Then there is a non-negative solution* $u \in W_0^{1,p}(\Omega)$ *of:*

$$\begin{cases} -\Delta_p u + |\nabla u|^p = g(x)u^{p-1} + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Remark 3.7. Observe that since $q < N$, we have:

$$(q^*)' < \frac{N}{q}$$

hence our results for the constant case $p = 2$ require less regularity of f than in [2, Theorem 2.4] to get existence in $W_0^{1,2}(\Omega)$. However, we impose more regularity on g than the used in [2]. We believe that the optimal regularity on g in all the above results should be:

$$g \in L^{(q^*(\cdot)/\eta(\cdot))'}(\Omega).$$

This remains open and will be treated in a future work.

As a concluding remark, we point out that the main results of the paper contribute to the fact that the presence of first-order terms produces regularization effects and permits the existence of solutions. In fact, suppose that for each $f \in L^1(\Omega)$ there is a weak (energy) solution $u \in W_0^{1,p(\cdot)}(\Omega)$ to:

$$-\Delta_{p(x)}u = u^{p(x)-1} + f(x) \quad \text{in } \Omega.$$

Hence:

$$L^1(\Omega) \subset W^{-1,p'(\cdot)}(\Omega)$$

which is a contradiction.

4. Proof of Theorem 3.2

4.1. **Previous results.** In this section we give preliminary results in order to prove Theorem 3.2 in the next section.

Given a non-negative measurable function u, we will consider the usual k-truncation functions T_k and G_k defined as:

$$T_k(u) := \begin{cases} u, & \text{if } |u| \leq k, \\ k, & \text{if } |u| \geq k. \end{cases}$$

and:

$$G_k(u) := u - T_k(u).$$

Observe that $G_k(u) = 0$ when $u \leq k$.

We start by proving the following technical result.

Lemma 4.1. *Let $0 < q(\cdot) < p(\cdot)$. Then for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ so that:*

$$s^{q(x)} \leq \varepsilon s^{p(x)} + C_\varepsilon, \quad \textit{for all } s \geq 0 \ \textit{ and } x \in \overline{\Omega}. \tag{4.1}$$

Proof. For $\varepsilon < 1$, by Young Inequality we observe that

$$\varepsilon s^{q(x)} \frac{1}{\varepsilon} \leq \frac{\varepsilon^{\frac{p(x)}{q(x)}}}{\frac{p(x)}{q(x)}} s^{p(x)} + \frac{\frac{1}{\varepsilon}^{r(x)}}{r(x)}$$

where $r(x) = \left(\frac{p(x)}{q(x)}\right)'$, using that $\frac{q(x)}{p(x)} \leq 1$ and $\varepsilon^{\frac{p(x)}{q(x)}} < \varepsilon$, we obtain that

$$s^{q(x)} \leq \varepsilon s^{p(x)} + C_\varepsilon$$

where $C_\varepsilon = \frac{\frac{1}{\varepsilon}^{r^+}}{r^-}$. Finally, for $\varepsilon \geq 1$, it is easy to see that

$$s^{q(x)} \leq \varepsilon s^{p(x)} + 1,$$

as we want to prove. ■

The following proposition gives the existence of solutions to Problem (1.1) for truncated zero-order terms and bounded data.

Proposition 4.2. *Let $f, g \in L^\infty(\Omega)$ be non-negative and let k be positive. Then there exists a non-negative solution $u_k \in W_0^{1,p(\cdot)}(\Omega)$ to the following equation:*

$$-\Delta_{p(x)} u + |\nabla u|^{q(x)} = g(x)(T_k u)^{\eta(x)} + f(x) \quad \textit{in } \Omega. \tag{4.2}$$

Proof. Let $v_k \in W_0^{1,p(\cdot)}(\Omega)$ be so that:

$$-\Delta_{p(x)} v_k = g(x) k^{\eta^+} + f(x). \tag{4.3}$$

Observe that $v_k \in L^\infty(\Omega)$ (for instance, by Corollary 3.2 in [22]). For each n consider the problem:

$$\begin{cases} -\Delta_{p(x)} w + G_n(x, w, \nabla w) = f(x), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

where for $(x, r, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$:

$$G_n(x, r, \xi) = \begin{cases} \chi_{[0,\infty)}(r) H_n(x, \xi) - g(x) T_k(r)^{\eta(x)} & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases}$$

where:

$$H_n(x, \xi) = \frac{|\xi|^{q(x)}}{1 + \frac{1}{n}|\xi|^{q(x)}}.$$

Observe that G_n is a Carathéodory function. By [1, Theorem 4.1], there is a solution $w_n \in W_0^{1,p(\cdot)}(\Omega)$ to (4.4). We shall prove that $w_n \geq 0$ for all n. We start by considering truncations of $(-w_n)^+$ for each $M \geq 0$:

$$(-w_n)^+_M = \begin{cases} (-w_n)^+ & \text{if } (-w_n)^+(x) \leq M \\ M & \text{if } (-w_n)^+(x) > M \end{cases}$$

Also, we define the following auxiliary sets:

$$\omega_0 = \{x \in \Omega : -w_n(x) \geq 0\}$$

$$\omega_0^M = \{x \in \Omega : 0 \leq -w_n(x) \leq M\}$$

It is clear that:

$$\begin{cases} (-w_n)^+_M = 0, & \text{if } x \in \Omega - \omega_0 \\ \nabla(-w_n)^+_M = 0, & \text{if } x \in \Omega - \omega_0^M. \end{cases}$$

As a result, using $(-w_n)^+_M$ as a test function in (4.4), we obtain:

$$\begin{aligned} 0 &\leq \int_\Omega f(x)(-w_n)^+_M \, dx \\ &= \int_\Omega |\nabla w_n|^{p(x)-2} \nabla w_n \cdot \nabla(-w_n)^+_M \, dx + \int_\Omega G_n(x, w_n, \nabla w_n)(-w_n)^+_M \, dx \\ &= -\int_{\omega_0^M} |\nabla w_n|^{p(x)} \, dx + \int_{\omega_0} G_n(x, 0, 0)(-w_n)_M \, dx \\ &= -\int_{\omega_0^M} |\nabla w_n|^{p(x)} \, dx. \end{aligned} \tag{4.5}$$

Thus for all $M \geq 0$:

$$\nabla(-w_n)^+ = 0 \quad a.e. \text{ in } \omega_0^M.$$

It follows that $\nabla(-w_n)^+ = 0$ a.e. in Ω and hence, since $(-w_n)^+ \in W_0^{1,p(\cdot)}(\Omega)$, $(-w_n)^+ = 0$ a.e. Hence, $w_n \geq 0$. Observe that

$$-\Delta_{p(x)} w \leq f(x) + g(x) k^{\eta^+} = -\Delta_{p(x)} v_k,$$

hence by comparison ([18, Lemma 2.2]) $w_n \leq v_k$ where v_k solves (4.3), and since w_n is non-negative, we get $\|w_n\|_{L^\infty(\Omega)} \leq \|v_k\|_{L^\infty(\Omega)}$ for all n. Thus w_n solves:

$$\begin{cases} -\Delta_{p(x)} w + H_n(x, \nabla w) = g(x)(T_k w)^{\eta(x)} + f(x), & \text{in } \Omega, \\ w = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

We study now the convergence of w_n. Using w_n as a test function in (4.6), we derive:

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p(x)}\, dx + \int_\Omega H_n(x, |\nabla w_n|) w_n\, dx \\ &\qquad = \int_\Omega g(x)(T_k w_n)^{\eta(x)} w_n\, dx + \int_\Omega f(x) w_n\, dx. \end{aligned}$$

Hence:

$$\int_\Omega |\nabla w_n|^{p(x)}\, dx \leq C(f, g, \Omega, k)$$

which implies, up to subsequence, that there is $u_k \in W_0^{1,p(\cdot)}(\Omega)$ so that $w_n \rightharpoonup u_k$ in $W_0^{1,p(\cdot)}(\Omega)$. By weak*-convergence in $L^\infty(\Omega)$ we derive $u_k \leq \|v_k\|_{L^\infty(\Omega)}$. We now prove that $w_n \to u_k$ strongly in $W_0^{1,p(\cdot)}(\Omega)$.

Consider $\phi(s) = s \exp\left(\frac{1}{4}s^2\right)$, which satisfies:

$$\phi'(s) - |\phi(s)| \geq \frac{1}{2}. \tag{4.7}$$

We use $\phi_n = \phi(w_n - u_k)$ as a test function in (4.6) and we obtain (we write $\phi_n' = \phi'(w_n - u_k)$):

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p(x)-2} \nabla w_n \cdot \nabla(w_n - u_k) \phi_n'\, dx + \int_\Omega H_n(x, \nabla w_n) \phi_n\, dx \\ &\qquad = \int_\Omega \Big(g(x)[T_k w_n]^{\eta(x)} \phi_n\, dx + f(x)\phi_n \Big)\, dx. \end{aligned} \tag{4.8}$$

Since ϕ_n is uniformly bounded and tends to 0 as $n \to \infty$, we conclude by Lebesgue Dominated Theorem that the right hand side of (4.8) tends to 0. Next, by Lemma 4.1 it follows:

$$\begin{aligned} &\Big| \int_\Omega \frac{|\nabla w_n|^{q(x)}}{1 + \frac{1}{n}|\nabla w_n|^{q(x)}} \phi_n\, dx \Big| \leq \varepsilon \int_\Omega |\nabla w_n|^{p(x)} |\phi_n|\, dx + C_\varepsilon \int_\Omega |\phi_n|\, dx \\ &\quad \leq \varepsilon 2^{p^+ - 1} \left(\int_\Omega |\nabla w_n - \nabla u_k|^{p(x)} |\phi_n|\, dx + \int_\Omega |\nabla u_k|^{p(x)} |\phi_n|\, dx \right) + C_\varepsilon \int_\Omega |\phi_n|\, dx. \end{aligned} \tag{4.9}$$

Again by Lebesgue's Theorem, the last two terms converge to 0 as $n \to \infty$. The first term in (4.8) is treated as follows:

$$\begin{aligned}
&\int_\Omega |\nabla w_n|^{p(x)-2}\nabla w_n \cdot \nabla(w_n - u_k)\phi_n'\,dx \\
&= \int_\Omega (|\nabla w_n|^{p(x)-2}\nabla w_n - |\nabla u_k|^{p(x)-2}\nabla u_k)\cdot\nabla(w_n-u_k)\phi_n'\,dx \\
&+ \int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k\cdot\nabla(w_n-u_k)\phi_n'\,dx.
\end{aligned} \tag{4.10}$$

Since ϕ_n' is bounded, $|\nabla u_k|^{p(\cdot)-2}\nabla u_k \in L^{p'(\cdot)}(\Omega)$ and $\nabla(w_n - u_k) \rightharpoonup 0$ in $L^{p(\cdot)}(\Omega)$ we derive by Lemma 2.8 that:

$$\lim_{n\to\infty}\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k\cdot\nabla(w_n-u_k)\phi_n'\,dx = 0.$$

We will use the well-known vector inequalities:

$$(|\xi|^{p(\cdot)-2}\xi - |\eta|^{p(\cdot)-2}\eta)\cdot(\xi-\eta) \ge \left(\frac{1}{2}\right)^{p(\cdot)} |\xi-\eta|^{p(\cdot)} \text{ if } p(\cdot)\ge 2.$$

$$(|\xi|^{p(\cdot)-2}\xi - |\eta|^{p(\cdot)-2}\eta)\cdot(\xi-\eta) \ge (p(\cdot)-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p(\cdot)}} \text{ if } 1 < p(\cdot) < 2.$$

We introduce the sets:

$$\Omega_1 = \{x\in\Omega : p(x)\ge 2\}$$

and:

$$\Omega_2 = \{x\in\Omega : p(x) < 2\}.$$

Now:

$$\int_\Omega |\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx = \int_{\Omega_1}|\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx + \int_{\Omega_2}|\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx.$$

We treat first the degenerate case:

$$\begin{aligned}
&\int_{\Omega_1}|\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx \\
&\le 2^{p^+}\int_{\Omega_1}(|\nabla w_n|^{p(x)-2}\nabla w_n - |\nabla u_k|^{p(x)-2}\nabla u_k)\cdot\nabla(w_n-u_k)\phi_n'\,dx \quad (\text{since } \phi_n' > 0) \\
&\le 2^{p^+}\int_{\Omega}(|\nabla w_n|^{p(x)-2}\nabla w_n - |\nabla u_k|^{p(x)-2}\nabla u_k)\cdot\nabla(w_n-u_k)\phi_n'\,dx \\
&\le 2^{p^+}\int_\Omega |\nabla w_n|^{p(x)-2}\nabla w_n\cdot\nabla(w_n-u_k)\phi_n'\,dx + o(1) \qquad (\text{by } (4.10)) \\
&\le 2^{2p^+-1}\varepsilon\int_\Omega |\nabla w_n - \nabla u_k|^{p(x)}|\phi_n|\,dx + o(1) \qquad (\text{by (4.9) and (4.8)}).
\end{aligned} \tag{4.11}$$

The uniform boundedness of w_n in $W_0^{1,p(\cdot)}(\Omega)$ and of $|\phi_n|$ in $L^\infty(\Omega)$ imply by (4.11) that:

$$\limsup_{n\to\infty}\int_{\Omega_1}|\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx \le C\varepsilon. \tag{4.12}$$

Next, writing:

$$\int_{\Omega_2} |\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx$$
$$=\int_{\Omega_2}\frac{|\nabla(w_n-u_k)|^{p(x)}(\phi_n')^{\frac{p(x)}{2}}}{(|\nabla w_n|+|\nabla u_k|)^{\frac{(2-p(x))p(x)}{2}}}(\phi_n')^{1-\frac{p(x)}{2}}(|\nabla w_n|+|\nabla u_k|)^{\frac{(2-p(x))p(x)}{2}}\,dx,$$

we obtain by Hölder's inequality and Lemma 2.3, that:

(4.13)

$$\frac{1}{2}\int_{\Omega_2} |\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx$$
$$\leq C\left\|\frac{|\nabla(w_n-u_k)|^{p(x)}(\phi_n')^{\frac{p(x)}{2}}}{(|\nabla w_n|+|\nabla u_k|)^{\frac{(2-p(x))p(x)}{2}}}\right\|_{L^{\frac{2}{p(\cdot)}}(\Omega_2)}\cdot\left\|(\phi_n')^{1-\frac{p(x)}{2}}(|\nabla w_n|+|\nabla u_k|)^{\frac{(2-p(x))p(x)}{2}}\right\|_{L^{\frac{2}{2-p(\cdot)}}(\Omega_2)}$$
$$\leq C\max\left\{\left(\int_\Omega\frac{|\nabla(w_n-u_k)|^2\phi_n'}{(|\nabla w_n|+|\nabla u_k|)^{2-p(x))}}\right)^{2/p^+},\left(\int_\Omega\frac{|\nabla(w_n-u_k)|^2\phi_n'}{(|\nabla w_n|+|\nabla u_k|)^{2-p(x))}}\right)^{2/p^-}\right\}$$
$$\leq C\max\left\{\left(\int_\Omega(|\nabla w_n|^{p(x)-2}\nabla w_n-|\nabla u_k|^{p(x)-2}\nabla u_k)\cdot\nabla(w_n-u_k)\phi_n'\,dx\right)^{2/p^+},(\cdots)^{2/p^-}\right\}$$
$$\leq C\max\left\{\left(\varepsilon\int_\Omega|\nabla w_n-\nabla u_k|^{p(x)}|\phi_n|\,dx\right)^{2/p^+},\left(\varepsilon\int_\Omega|\nabla w_n-\nabla u_k|^{p(x)}|\phi_n|\,dx\right)^{2/p^-}\right\}+o(1),$$

where we have used (4.10) and (4.8). Using again the boundedness of w_n, u_k and $|\phi_n|$ we have by (4.13) that:

$$\limsup_{n\to\infty}\int_{\Omega_2}|\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx\leq C\max\left\{\varepsilon^{2/p^+},\varepsilon^{2/p^-}\right\}. \tag{4.14}$$

Combining (4.12) and (4.14), observing that $\phi_n'\geq 1$ and letting $\varepsilon\to 0$, we conclude the strong convergence of w_n to u_k in $W_0^{1,p(\cdot)}(\Omega)$.

Hence for any $\phi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$:

- $\int_\Omega|\nabla w_n|^{p(x)-2}\nabla w_n\cdot\nabla\phi\,dx\to\int_\Omega|\nabla u_k|^{p(x)-2}\nabla u_k\cdot\nabla\phi\,dx$ since the term
$$|\nabla w_n|^{p(\cdot)-2}\nabla w_n$$
is bounded in $L^{p'(\cdot)}(\Omega)$ and $|\nabla w_n|^{p(x)-2}\nabla w_n\to|\nabla u_k|^{p(x)-2}\nabla u_k$ a.e. in Ω, so we may apply Lemma 2.7.

- $\int_\Omega H_n(x,\nabla w_n)\phi\,dx\to\int_\Omega|\nabla u_k|^{q(x)}\phi\,dx$ again by Lemma 2.7 since
$$H_n(x,\nabla w_n)\to|\nabla u_k|^{q(x)}$$
a.e. in Ω and $H_n(x,\nabla w_n)$ is bounded in $L^{p(\cdot)/q(\cdot)}(\Omega)$.

- $\int_\Omega g(x)(T_k(w_n))^{\eta(x)}\phi\,dx\to\int_\Omega g(x)(T_k(u_k))^{\eta(x)}\phi\,dx$ by Lebesgue's Theorem.

Therefore, u_k solves (4.2).

■

We are now in position to prove Theorem 3.2.

4.2. **Proof of Theorem 3.2.** For each n, let $g_n = T_n(g)$ and $f_n = T_n(f)$. By Proposition 4.2 there is $u_n \in W_0^{1,p(\cdot)}(\Omega)$, non-negative, so that:

$$\left\{ \begin{array}{c} -\Delta_{p(x)} u_n + |\nabla u_n|^{q(x)} = g_n(x)(T_n u_n)^{\eta(x)} + f_n(x), \quad \text{in } \Omega, \\ u_n = 0, \quad \text{on } \partial\Omega. \end{array} \right. \tag{4.15}$$

We start assuming that $\|\nabla u_n\|_{L^{q(\cdot)}(\Omega)} \geq 1$ for all n. Taking $T_k(u_n)$ as a test function in (4.15) we derive:

$$\begin{aligned} &\int_\Omega |\nabla T_k u_n|^{p(x)}\,dx + \int_\Omega |\nabla u_n|^{q(x)} T_k u_n\,dx \\ &\quad = \int_\Omega g_n(x)(T_n u_n)^{\eta(x)} T_k u_n\,dx + \int_\Omega f_n(x) T_k u_n\,dx \\ &\quad \leq k\left(\int_\Omega g_n(x) u_n^{\eta(x)}\,dx\right) + k\|f_n\|_{L^1(\Omega)}. \end{aligned} \tag{4.16}$$

In the case $\int_\Omega g(x) u_n^{\eta(x)}\,dx \leq 1$ we have:

$$\int_\Omega |\nabla T_k u_n|^{p(x)}\,dx + \int_\Omega |\nabla u_n|^{q(x)} T_k u_n\,dx \leq k\|f\|_{L^1(\Omega)} \tag{4.17}$$

and when $\int_\Omega g(x) u_n^{\eta(x)}\,dx > 1$ by Young's inequality, Proposition 2.2 and (3.2) we obtain:

(4.18)

$$\begin{aligned} &\int_\Omega |\nabla T_k u_n|^{p(x)}\,dx + \int_\Omega |\nabla u_n|^{q(x)} T_k u_n\,dx \leq k\left(\int_\Omega g_n(x) u_n^{\eta(x)}\,dx\right) + k\|f_n\|_{L^1(\Omega)} \\ &\leq \frac{\varepsilon k^{q^-/\eta^+}}{q^-/\eta^+}\left(\int_\Omega g_n(x) u_n^{\eta(x)}\,dx\right)^{q^-/\eta^+} + C(\varepsilon) + k||f||_{L^1(\Omega)} \\ &\leq \frac{\varepsilon k^{q^-/\eta^+}}{q^-/\eta^+}\|g_n^{1/\eta(\cdot)} u_n\|^{q^-}_{L^{\eta(\cdot)}(\Omega)} + C(\varepsilon) + k||f||_{L^1(\Omega)} \\ &\leq \frac{\varepsilon k^{q^-/\eta^+}}{C(g,\eta,q)q^-/\eta^+}\|\nabla u_n\|^{q^-}_{L^{q(\cdot)}(\Omega)} + C(\varepsilon) + k||f||_{L^1(\Omega)}. \end{aligned}$$

Hence:

(4.19)

$$\begin{aligned} &\|\nabla u_n\|^{q^-}_{L^{q(\cdot)}(\Omega)} \leq \int_\Omega |\nabla u_n|^{q(x)}\,dx \\ &\leq \int_\Omega |\nabla T_k u_n|^{q(x)}\,dx + k\int_{\{u_n \geq k\}} |\nabla u_n|^{q(x)}\,dx \\ &\leq \int_\Omega |\nabla T_k u_n|^{p(x)}\,dx + \int_{\{u_n \geq k\}} |\nabla u_n|^{q(x)} T_k u_n\,dx + |\Omega| \quad \text{(by Young's inequality)} \\ &\leq \max\left\{k\|f\|_{L^1(\Omega)}, \frac{\varepsilon k^{q^-/\eta^+}\eta^+}{C(g,\eta,q)q^-}\|\nabla u_n\|^{q^-}_{L^{q(\cdot)}} + C(\varepsilon) + k||f||_{L^1(\Omega)} + |\Omega|\right\} \end{aligned}$$

where we have used (4.17) and (4.18). Choosing ε small, we derive $\|\nabla u_n\|_{L^{q(\cdot)}(\Omega)} \leq C$. Thus up to a subsequence:

- $u_n \rightharpoonup u$ in $W_0^{1,q(\cdot)}(\Omega)$;
- $T_k u_n \rightharpoonup T_k u$ in $W_0^{1,p(\cdot)}(\Omega)$;
- $u_n \to u$ in $L^{s(\cdot)}(\Omega)$, for $s(\cdot) < q^*(\cdot)$.

If $\|\nabla u_n\|_{L^{q(\cdot)}(\Omega)} \leq 1$ for a subsequence, we obtain the same conclusions. Using $\psi_{k-1}(u_n) = T_1(G_{k-1}(u_n))$ as a test function in (4.15) we derive:

$$\begin{aligned} &\int_\Omega |\nabla \psi_{k-1}(u_n)|^{p(x)}\,dx + \int_\Omega \psi_{k-1}(u_n)|\nabla u_n|^{q(x)}\,dx \\ &\qquad = \int_\Omega \Big(g_n(x)(T_n u_n)^{\eta(x)} + f_n(x)\Big)\,\psi_{k-1}(u_n)\,dx. \end{aligned} \tag{4.20}$$

The last integral may be divided as:

$$\begin{aligned} &\int_{\{u_n \geq k\}} \Big(g_n(x)(T_n u_n)^{\eta(x)} + f_n(x)\Big)\,\psi_{k-1}(u_n)\,dx \\ &\qquad \int_{\{k-1 \leq u_n \leq k\}} \Big(g_n(x)(T_n u_n)^{\eta(x)} + f_n(x)\Big)\,\psi_{k-1}(u_n)\,dx, \end{aligned} \tag{4.21}$$

since $\psi_{k-1}(u_n) = 0$ if $u_n \leq k-1$. Moreover, since u_n is uniformly bounded in $L^1(\Omega)$ we derive by Chebyshev's inequality that:

$$|\,\{x \in \Omega : k \leq u_n\}\,| \to 0 \tag{4.22}$$

uniformly in n as $k \to \infty$. By the definition of ψ_{k-1} and Hölder's inequality we have:

$$\begin{aligned} &\int_{\{u_n \geq k\}} \Big(g_n(x)(T_n u_n)^{\eta(x)} + f_n(x)\Big)\,\psi_{k-1}(u_n)\,dx \\ &\qquad + \int_{\{k-1 \leq u_n \leq k\}} \Big(g_n(x)(T_n u_n)^{\eta(x)} + f_n(x)\Big)\,\psi_{k-1}(u_n)\,dx \\ &\leq \int_{\{u_n \geq k-1\}} \Big(g(x)u_n^{\eta(x)} + f(x)\Big)\,dx \\ &\leq \Big(\|g\|_{L^{(q^*(\cdot)/\eta(\cdot))'}(\{u_n \geq k-1\})}\|u_n^{\eta(\cdot)}\|_{L^{q^*(\cdot)/\eta(\cdot)}(\Omega)} + \|f\|_{L^1(\{u_n \geq k-1\})}\Big) \\ &\leq \max\left\{\left(\int_{\{k-1 \leq u_n\}} g(x)^{\left[\frac{q^*(x)}{\eta(x)}\right]'}\,dx\right)^{1/\gamma^-}, \left(\int_{\{k-1 \leq u_n\}} g(x)^{\left[\frac{q^*(x)}{\eta(x)}\right]'}\,dx\right)^{1/\gamma^+}\right\} \|u_n^{\eta(\cdot)}\|_{L^{\frac{q^*}{\eta}}(\Omega)} \\ &\qquad\qquad + \|f\|_{L^1(\{u_n \geq k-1\})}. \end{aligned} \tag{4.23}$$

where:

$$\gamma(\cdot) = \left(\frac{q^*(\cdot)}{\eta(\cdot)}\right)'.$$

Now, by the weak convergence of u_n to u in $W_0^{1,q(\cdot)}(\Omega)$, there is $C > 1$ so that:

$$\int_\Omega u_n^{q^*(\cdot)}\,dx \leq C.$$

Hence, $u_n^{\eta(\cdot)}$ is bounded in $L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$. Moreover, by (4.22):

$$\max\left\{\left(\int_{\{k-1\le u_n\}} g(x)^{[q^*(x)/\eta(x)]'}\,dx\right)^{1/\gamma^-},\left(\int_{\{k-1\le u_n\}} g(x)^{[q^*(x)/\eta(x)]'}\,dx\right)^{1/\gamma^+}\right\} + \|f\|_{L^1(\{u_n\ge k-1\})}$$

goes to 0 as $k\to\infty$, uniformly in n. Thus:

$$\int_{\{u_n\ge k\}}\left(g_n(x)u_n^{\eta(x)}+f_n(x)\right)\psi_{k-1}(u_n)\,dx + \int_{\{k-1\le u_n\le k\}}\left(g_n(x)u_n^{\eta(x)}+f_n(x)\right)\psi_{k-1}(u_n)\,dx\to 0$$

as $k\to\infty$ uniformly in n. It follows that:

$$\lim_{k\to\infty}\int_{\{u_n\ge k\}}|\nabla u_n|^{q(x)}\,dx=0,\qquad \text{uniformly in } n. \tag{4.24}$$

Now we want to prove that for each fix k we have:

$$T_k u_n\to T_k u\quad \text{strongly in } W_0^{1,q(\cdot)}(\Omega).$$

Take $v_n=\phi(T_k(u_n)-T_k(u))$ as a test function in (4.15) (where ϕ satisfies (4.7)). We get:

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(T_k(u_n)-T_k(u))\phi_n'\,dx+\int_\Omega|\nabla u_n|^{q(x)}v_n\,dx\\ =\int_\Omega f_n(x)v_n\,dx+\int_\Omega g_n(x)(T_nu_n)^{\eta(x)}v_n\,dx,\end{aligned}\tag{4.25}$$

with $\phi_n'=\phi'(T_k(u_n)-T_k(u))$. Firstly, the term:

$$\int_\Omega f_n(x)v_n\,dx\to 0 \text{ as } n\to\infty \tag{4.26}$$

by Lebesgue's Theorem. Now we treat the term:

$$\int_\Omega g_n(x)(T_nu_n)^{\eta(x)}v_n\,dx.$$

Since $u_n^{\eta(\cdot)}$ is bounded in $L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$, there is $w\in L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$ so, up to subsequence, that:

$$u_n^{\eta(\cdot)}\rightharpoonup w\quad \text{in } L^{q^*(\cdot)/\eta(\cdot)}(\Omega). \tag{4.27}$$

Since we also have $u_n^\eta\to u^\eta$ a.e., we conclude that $w=u^{\eta(\cdot)}$ by Lemma 2.7. By Egorov's Theorem, for each ε there is a measurable set A_ε so that $|A_\varepsilon|<\varepsilon$ and T_ku_n

converges to $T_k u$ uniformly in $\Omega \setminus A_\varepsilon$. Then:

$$\begin{aligned}\int_\Omega g_n(x)(T_n u_n)^{\eta(x)} v_n\,dx &= \int_{\Omega\setminus A_j} g_n(T_n u_n)^{\eta(x)}\left[\phi(T_k u_n - T_k u)\right]dx\\ &+\int_{A_j} g_n(x)(T_n u_n)^{\eta(x)}\left[\phi(T_k u_n - T_k u)\right]dx\\ &\le o(1)\int_\Omega g(x)u_n^{\eta(x)}\,dx + \phi(2k)\int_{A_j} g(x)u_n^{\eta(x)}\,dx\end{aligned}$$

When $n \to \infty$, the first term in the last equality tends to 0 (by (4.27), the fact that $g \in L^{(q^*(\cdot)/\eta(\cdot))'}(\Omega)$ and the uniform convergence of $T_k u_n$ to $T_k u$) and the last term converges to:

$$\phi(2k)\int_{A_j} g(x)u^{\eta(x)}\,dx$$

which can be arbitrarily small. Thus:

$$\int_\Omega g_n(x)(T_n u_n)^{\eta(x)} v_n\,dx \to 0. \tag{4.28}$$

In (4.25) we decompose:

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx$$

as the sum:

$$\begin{aligned}&\int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx\\ &+\int_\Omega |\nabla G_k u_n|^{p(x)-2}\nabla G_k u_n \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx.\end{aligned} \tag{4.29}$$

Since $G_k(u_n) = 0$ in $\{u_n \le k\}$, we have that the last term in (4.29) equals:

$$-\int_\Omega |\nabla G_k(u_n)|^{p(x)-2}\nabla G_k(u_n)\cdot \nabla T_k(u)\chi_{\{u_n\ge k\}}\phi_n'\,dx. \tag{4.30}$$

Observe that:

$$\nabla T_k(u)\chi_{\{u_n\ge k\}}\phi_n' \to 0$$

a.e. in Ω and by Lebesgue's Theorem, the convergence is in $L^{r(\cdot)}(\Omega)$ for all $r(\cdot) \le p(\cdot)$. Now we shall prove that there is $C > 0$ so that[1]:

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)}\,dx \le C \quad \text{for all } n. \tag{4.31}$$

Observe that (4.31) and the boundedness of $T_k u_n$ imply that $u \in W_0^{1,p(\cdot)}(\Omega)$ since:

$$\int_\Omega |\nabla u_n|^{p(x)}dx \le \int_\Omega |\nabla T_k(u_n)|^{p(x)}dx + \int_\Omega |\nabla G_k(u_n)|^{p(x)}dx \le C$$

[1]Observe that the boundedness of $\nabla G_k(u_n)$ holds automatically when $p = q$ by (4.24), that is the case in [7].

for some $C > 0$. Next, to prove (4.31), take $G_k(u_n)$ as a test function in (4.15) we derive:

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)}\,dx = \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla G_k(u_n)\,dx$$
$$\le \int_\Omega g_n(x)T_n(u_n)^{\eta(\cdot)}G_k(u_n)\,dx + \int_\Omega f_n(x)G_k(u_n)\,dx.$$

The uniform boundedness follows by the assumptions on g and f (see the conditions on the exponents (3.1)) and the fact that u_n is uniformly bounded in $L^{q^*(\cdot)}(\Omega)$. Hence:

$$|\nabla G_k(u_n)|^{p(x)-2}\nabla G_k(u_n)$$

is uniformly bounded in $L^{p'(\cdot)}(\Omega)$ for large n and thus (4.30) is of order $o(1)$.

The first term in (4.29) is re-writing as:

$$\begin{aligned}
&\int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx \\
&= \int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big) \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx \\
&+ \int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx
\end{aligned} \tag{4.32}$$

The last term in (4.32) tends to 0 as $n \to \infty$ by Lemma 2.8. Summarizing, from (4.25), (4.26), (4.28), (4.29) and (4.32), we obtain:

$$\begin{aligned}
0 &\le \int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big) \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx \\
&= -\int_\Omega |\nabla u_n|^{q(x)} v_n\,dx + o(1) \\
&= -\int_{\{u_n<k\}} |\nabla u_n|^{q(x)} v_n\,dx - \int_{\{u_n\ge k\}} |\nabla u_n|^{q(x)} v_n\,dx + o(1) \\
&\le -\int_{\{u_n<k\}} |\nabla u_n|^{q(x)} v_n\,dx + o(1).
\end{aligned} \tag{4.33}$$

Observe that:

$$\int_{\{u_n<k\}} |\nabla u_n|^{q(x)} v_n\,dx = \int_{\{u_n<k\}} |\nabla T_k u_n|^{q(x)} v_n\,dx = \int_\Omega |\nabla T_k u_n|^{q(x)} v_n\,dx. \tag{4.34}$$

Since $|\nabla T_k u_n|^{q(x)}$ is bounded in $L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)$ and v_n is uniformly bounded and converges pointwise to 0, we derive, up to subsequence, that $|\nabla T_k u_n|^{q(x)} v_n \rightharpoonup 0$ in $L^{\frac{p(\cdot)}{q(\cdot)}}(\Omega)$, by Lemma 2.7. Therefore:

$$\int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big) \cdot \nabla(T_k(u_n) - T_k(u))\phi_n'\,dx = o(1).$$

By Theorem 2.6, we derive the strong convergence of $T_k u_n$ to $T_k u$ in $W_0^{1,p(\cdot)}(\Omega)$, and hence in $W_0^{1,q(\cdot)}(\Omega)$.

Finally, for any $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we shall prove that:

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\, dx + \int_\Omega |\nabla u_n|^{q(x)}\varphi\, dx \\ = \int_\Omega g_n(x)(T_n u_n)^{\eta(x)}\varphi\, dx + \int_\Omega f(x)\varphi\, dx\end{aligned} \tag{4.35}$$

converges to:

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx + \int_\Omega |\nabla u|^{q(x)}\varphi\, dx = \int_\Omega g(x)u^{\eta(x)}\varphi\, dx + \int_\Omega f(x)\varphi\, dx.$$

For the convergence of the first term we proceed as follows:

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\, dx = \int_{\{u_n\geq k\}} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\, dx \\ + \int_{\{u_n\leq k\}} |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla\varphi\, dx.\end{aligned}$$

For the last term we have the facts (consequences of the strong convergence of $T_k u_n$ to $T_k u$):

(1) $|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla\varphi\chi_{\{u_n\leq k\}} \to |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla\varphi\chi_{\{u\leq k\}}$ a.e. in Ω.

(2) $|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n$ is bounded in $L^{p'(\cdot)}(\Omega)$.

Hence, by Lemma 2.7:

$$\lim_{n\to\infty}\int_{\{u_n\leq k\}} |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla\varphi\, dx = \int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla\varphi\, dx.$$

Thus, by (4.24) and the assumption $p(\cdot) - 1 \leq q(\cdot)$, we derive:

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\, dx = \int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla\varphi\, dx + o(1), \text{ as } k\to\infty. \tag{4.36}$$

Recalling that $u \in W_0^{1,p(\cdot)}(\Omega)$, it follows that $|\nabla T_k u|^{p(x)-2}\nabla T_k u$ is bounded in $L^{p'(\cdot)}(\Omega)$, hence making $k \to \infty$ in (4.36) and appealing again to Lemma 2.7 it follows the desired convergence.

Next, we deal the second term in (4.35). Indeed, we will derive that $|\nabla u_n|^{q(x)} \to |\nabla u|^{q(x)}$ strongly in $L^1(\Omega)$ by appealing to Vitali's Lemma. First, we show that $|\nabla u_n|^{q(\cdot)}$ is uniformly integrable. Indeed, let $\varepsilon > 0$. By (4.24), there is k so that:

$$\int_{\{u_n\geq k\}} |\nabla u_n|^{q(x)}\, dx < \frac{\varepsilon}{3} \quad \text{for all } n. \tag{4.37}$$

Let now $\delta_0 > 0$ be so that for any measurable set E with $|E| < \delta_0$, there holds:

$$\int_E |\nabla T_k u|^{q(x)}\, dx < \frac{\varepsilon}{3}. \tag{4.38}$$

By the strong convergence of $T_k u_n$ to $T_k u$ in $W_0^{1,q(\cdot)}(\Omega)$ we derive that there is N (depending on ε and k) so that $n \geq N$ implies for any $|E| < \delta_0$:

$$\int_E |\nabla T_k u_n|^{q(x)}\,dx < \frac{\varepsilon}{3} + \int_E |\nabla T_k u|^{q(x)}\,dx < \frac{2\varepsilon}{3} \tag{4.39}$$

in view of (4.38). Thus, for any $n \geq N$ and any set $|E| < \delta_0$ we have by (4.37) and (4.39) that:

$$\int_E |\nabla u_n|^{q(x)}\,dx \leq \int_{\{u_n \geq k\}\cap E} |\nabla u_n|^{q(x)}\,dx + \int_E |\nabla T_k u_n|^{q(x)}\,dx < \varepsilon.$$

Moreover, for any $i \in \{1, ..., N-1\}$, there is $\delta_i > 0$ so that for any $|E| < \delta_i$:

$$\int_E |\nabla u_i|^{q(x)}\,dx < \varepsilon, \quad i = 1, ..., N-1.$$

Therefore, the uniform integrability follows by choosing $\delta = \min\{\delta_0, \delta_1, ..., \delta_{N-1}\}$. We also observe that, by the strong convergence of truncates, $|\nabla u_n|^{q(x)} \to |\nabla u|^{q(x)}$ a. e. in Ω. Hence, by Vitali's Convergence Theorem, we derive $|\nabla u_n|^{q(x)} \to |\nabla u|^{q(x)}$ strongly in $L^1(\Omega)$.

Finally, we treat the statement:

$$\int_\Omega g_n (T_n u_n)^{\eta(x)} \varphi\,dx \to \int_\Omega g u^{\eta(x)} \varphi\,dx \text{ as } n \to \infty. \tag{4.40}$$

Write:

$$\begin{aligned}
&\int_\Omega g u^{\eta(x)} \varphi\,dx - \int_\Omega g_n (T_n u_n)^{\eta(x)} \varphi\,dx \\
&= \int_\Omega g(u^{\eta(x)} - u_n^{\eta(x)})\varphi\,dx + \int_\Omega g[u_n^{\eta(x)} - (T_n u_n)^{\eta(x)}]\varphi\,dx \\
&\qquad + \int_\Omega (g - g_n)(T_n u_n)^{\eta(x)} \varphi\,dx.
\end{aligned}$$

Now:

- The convergence:

$$\int_\Omega g(u^{\eta(x)} - u_n^{\eta(x)})\varphi\,dx \to 0$$

  holds by the weak convergence of $u_n^{\eta(x)}$ to $u^{\eta(x)}$ in $L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$ and the assumptions on g.
- Observe that for a. e. x, $u_n(x) \to u(x)$ as $n \to \infty$. Hence, $T_n u_n(x) \to u(x)$ a. e. On the other hand,

$$\int_\Omega |u_n^{\eta(x)} - (T_n u_n)^{\eta(x)}|^{q^*(x)/\eta(x)} dx \leq \int_\Omega |u_n|^{q^*(x)} dx \leq C.$$

  Hence, by Lemma 2.7, $u_n^{\eta(x)} - (T_n u_n)^{\eta(x)} \rightharpoonup 0$ in $L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$, and consequently,

$$\int_\Omega g[u_n^{\eta(x)} - (T_n u_n)^{\eta(x)}]\varphi\,dx \to 0.$$

- Finally,
$$\int_\Omega (g-g_n)(T_n u_n)^{\eta(x)}\varphi\,dx \to 0$$
by Hölder's inequality, the convergence $g_n \to g$ in $L^{(q^*(\cdot)/\eta(\cdot))'}(\Omega)$ and the boundedness of $u_n^{\eta(\cdot)}$ in $L^{q^*(\cdot)/\eta(\cdot)}(\Omega)$.

This proves statement (4.40) and the proof of the theorem is finished.

5. Proof of Theorem 3.4

The proof mainly goes as for Theorem 3.2 for $p(\cdot)\geq 2$. We point out the differences. Firstly, we choose:

$$\phi = s\exp\left(2^{(4p^+-2)}s^2\right)$$

and (4.7) is now:

$$\phi' - 2^{2p^+-1}\phi \geq C > 0. \tag{5.1}$$

Next, (4.9) reads as:

$$\begin{aligned}\int_\Omega \frac{|\nabla w_n|^{p(x)}}{1+\frac{1}{n}|\nabla w_n|^{p(x)}}\phi_n dx &\leq \int_\Omega |\nabla w_n|^{p(x)}\phi_n dx\\ &\leq 2^{p^+-1}\int_\Omega |\nabla w_n - \nabla u_k|^{p(x)}\phi_n dx + o(1),\end{aligned} \tag{5.2}$$

and hence (4.11) yields:

$$\int_\Omega |\nabla(w_n-u_k)|^{p(x)}\phi_n'\,dx \leq 2^{2p^+-1}\int_\Omega |\nabla w_n - \nabla u_k|^{p(x)}|\phi_n|\,dx + o(1). \tag{5.3}$$

The strong converge of w_n to u_k in $W_0^{1,p(\cdot)}(\Omega)$ is obtained appealing to (5.1) and to (5.3). Moreover, since we are not allowed to use Lemma 2.7, the convergence $\int_\Omega H(x,\nabla w_n)\phi\,dx \to \int_\Omega |\nabla u_k|^{p(x)}\phi\,dx$ may be obtained as[2]:

$$\begin{aligned}&\Big|\int_\Omega \phi\left(H(x,\nabla w_n) - \frac{|\nabla u_k|^{p(x)}}{1+\frac{1}{n}|\nabla w_n|^{p(x)}} + \frac{|\nabla u_k|^{p(x)}}{1+\frac{1}{n}|\nabla w_n|^{p(x)}} - |\nabla u_k|^{p(x)}\right)dx\Big|\\ &\leq C\left(\int_\Omega \big||\nabla w_n|^{p(x)} - |\nabla u_k|^{p(x)}\big| + \int_\Omega\left(1-\frac{1}{1+\frac{1}{n}|\nabla w_n|^{p(x)}}\right)|\nabla u_k|^{p(x)}\,dx\right)\\ &= o(1),\end{aligned}$$

where we have used the strong convergence of w_n to u_k in $W_0^{1,p(\cdot)}(\Omega)$ and Lebesgue's Theorem for the last integral. Regarding the proof of Theorem 3.2, we first point

[2]Observe that this argument is also valid for $q(\cdot) < p(\cdot)$.

out that the boundedness (4.31) is obtained directly from (4.24). Moreover, the other part to be changed is (4.34), since we cannot use Lemma 2.7. Now, we write:

$$\begin{aligned}
&\int_\Omega |\nabla T_k u_n|^{p(x)} v_n\,dx \\
&\quad= \int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla T_k u_n\, v_n\,dx \\
&\qquad\qquad + \int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla (T_k u_n - T_k u) v_n\,dx \\
&\quad - \int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla (T_k u_n - T_k u) v_n\,dx \\
&\qquad\qquad + \int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla T_k u\, v_n\,dx \\
&\quad - \int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla T_k u\, v_n\,dx \\
&\quad= \int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big)\cdot \nabla (T_k u_n - T_k u) v_n\,dx \\
&\qquad + o(1),
\end{aligned}$$

where the terms:

$$\int_\Omega |\nabla T_k u|^{p(x)-2}\nabla T_k u \cdot \nabla (T_k u_n - T_k u) v_n\,dx$$

and:

$$\int_\Omega |\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n \cdot \nabla T_k u\, v_n\,dx$$

converge to 0 by Lemma 2.8. Hence, by (4.33), it follows:

$$\begin{aligned}
&\int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big)\cdot \nabla (T_k(u_n) - T_k(u))\phi_n'\,dx \\
&\quad\le \int_\Omega \Big(|\nabla T_k u_n|^{p(x)-2}\nabla T_k u_n - |\nabla T_k u|^{p(x)-2}\nabla T_k u\Big)\cdot \nabla (T_k u_n - T_k u)|v_n|\,dx \\
&\qquad + o(1).
\end{aligned}$$

Appealing to (5.1), we derive the strong convergence of ∇w_n to ∇u_k in $L^{p(\cdot)}(\Omega)$. The rest of the proof is the same as for Theorem 3.2.

Remark 5.1. Regarding the extension of Theorem 3.4 to all values of $p(x)$, we point out that in the singular framework, the absence of ε in (5.2) brings difficulties in order to deal with inequality (4.13) and hence to obtain the key control (4.14).

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