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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DIRICHLET PROBLEM IN FRACTIONAL ORLICZ-SOBOLEV SPACES

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ABSTRACT. In this paper, we first prove the existence of solutions to Dirichlet problems involving the fractional g-Laplacian operator and lower order terms by appealing to sub- and supersolution methods. Moreover, we also state the existence of extremal solutions. Afterwards, and under additional assumptions on the lower order structure, we establish by variational techniques the existence of multiple solutions: one positive, one negative and one with non-constant sign.

1. INTRODUCTION

Let us consider the following non local and non standard growth problem

$$\begin{cases} (-\triangle_g)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1}$$

where $s \in (0,1)$, Ω is a smooth and bounded domain in $\mathbb{R}^n$, and $(-\triangle_g)^s$ is the fractional g-Laplacian defined for sufficiently smooth functions u as

$$(-\triangle_g)^s u(x) = 2p.v \int_{\mathbb{R}^n} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right) \frac{dy}{|x-y|^{n+s}},$$

where $p.v.$ stands for 'in principal value' and $G' = g$ is an N-function. We recall the definition of N-function and its properties in Section 2. For a complete guide of this setting we mention the now classical book [17].

The fractional Orlicz-Sobolev spaces and their relation with the g-Laplacian operator were discussed in [14]. These spaces are the appropriate functional framework for $(-\triangle_g)^s$. For the reader convenience, we include some basic definitions and properties in Section 2.

Throughout the paper, we consider that f has a subcritical growth in the sense of the Orlicz-Sobolev embedding [2]. The main feature on the non linear term f is that no oddness condition is imposed.

As far as we know, the existence and multiplicity of solutions for equations involving the fractional g-Laplacian operator have been approached recently. Indeed, in [4], they provide existence of a non-negative solution for fractional g-Laplacian problems with a source $f(x,u)$ having a power-growth in u. Moreover, in [5] the existence of a nodal solution, that is, a changing-sing solution, was given. Regarding multiple solutions for $(-\triangle_g)^s$, we quote the work [7], for the existence of two

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non-trivial solutions, and [6], where infinitely many solutions are obtained for a class of non-local Orlicz-Sobolev Schrödinger equations.
Inspired by [11], we obtain an existence result using the sub-supersolution method. More precisely, we consider a sub-supersolution pair $(\underline{u}, \overline{u})$ and we define the set $S(\underline{u}, \overline{u})$ as follows

$$S(\underline{u}, \overline{u}) = \left\{ u \in W_0^{s,G}(\Omega) \colon u \text{ is a solution of (1), } \underline{u} \leq u \leq \overline{u} \right\}.$$

One of the main result of the article states that if $(\underline{u}, \overline{u})$ is a sub-supersolution pair of (1), then there exists $u \in S\,(\underline{u}, \overline{u})$, and hence (1) admits a solution.

Finally, we show, under some growth conditions in the source, the existence of three different nontrivial solutions for equation (1). More precisely, these solutions are one positive, one negative and one with non-constant sign. The method that we employ was introduced in [26] and was useful in different context, see for example [9, 25, 12, 18, 13] for the local case and [8] for the nonlocal context. This technique consists in restricting the functional associated to (1) to three different manifolds constructed by sign and normalization conditions. Unlike [26], here we do not appeal to the Ljusternik-Schnirelman theory. Instead, we use the Ekeland variational principle to prove the existence of a critical point of each restricted functional which turns out to be a critical point of the unrestricted functional and, consequently, is also a weak solution of (1).
The paper is organized as follows. In Section 2 we collect some basic facts of Orlicz-Sobolev spaces and the fractional g-Laplacian used throughout the manuscript. Section 3 is devoted to the study of the set $S\,(\underline{u}, \overline{u})$. Finally, in section 4 we prove the multiplicity of weak solutions for (1).

2. Preliminaries

2.1. **N-functions.** In this section we introduce basic definitions and preliminary results related to Orlicz spaces. For more details see [17]. We start recalling the definition of an N-function.

Definition 1. *A function $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an N-function if it admits the representation*

$$G(t) = \int_0^t g(\tau)d\tau,$$

where the function g is right-continuous for $t \geq 0$, positive for $t > 0$, non-decreasing, and satisfies the conditions

$$g(0) = 0, \quad g(\infty) = \lim_{t\to\infty} g(t) = \infty.$$

We extend G to $\mathbb{R}$ as an even function. By [17, Chapter 1, Sec. 5], an N-function has also the following properties:

(1) G is continuous, convex and even.
(2) G is super-linear at zero and at infinite, that is

$$\lim_{x\to 0} \frac{G(x)}{x} = 0$$

and

$$\lim_{x\to\infty} \frac{G(x)}{x} = \infty.$$

Indeed, the above conditions serve as an equivalent definition of N-functions.

An important property for N-functions is the following:

Definition 2. *We say that the N-function G satisfies the* $\triangle_2$ *condition if there exists* $C > 2$ *such that*

$$G(2x) \le CG(x) \ \textit{ for all } x \in \mathbb{R}_+.$$

According to [17, Theorem 4.1, Chapter 1] a necessary and sufficient condition for an N-function to satisfy the $\triangle_2$ condition is that there is $p^+ > 1$ such that

$$\frac{tg(t)}{G(t)} \le p^+, \ \forall\, t > 0. \tag{2}$$

Associate to G is the N-function complementary to it which is defined as follows:

$$\widetilde{G}(t) := \sup \{tw - G(w) \colon w > 0\}. \tag{3}$$

The definition of the complementary function assures that the following Young-type inequality holds

$$at \le G(t) + \widetilde{G}(a) \text{ for every } a, t \ge 0. \tag{4}$$

By [17, Theorem 4.3, Chapter 1], a necessary and sufficient condition for the N-function $\widetilde{G}$ complementary to G to satisfy the $\triangle_2$ condition is that there is $p^- > 1$ such that

$$p^- \le \frac{tg(t)}{G(t)}, \ \forall\, t > 0.$$

From now on, we will assume that the N-function $G(t) = \int_0^t g(\tau)d\tau$ satisfies the following growth behavior:

$$1 < p^- \le \frac{tg(t)}{G(t)} \le p^+ < \infty, \ \forall t > 0. \tag{5}$$

Hence, G and $\widetilde{G}$ both satisfy the $\triangle_2$ condition. Observe that (5) holds for all $t \in \mathbb{R}$, $t \neq 0$. Another useful condition (which implies (5)) is the following:

$$p^- - 1 \le \frac{tg'(t)}{g(t)} \le p^+ - 1, \ \forall t > 0. \tag{6}$$

Again, (6) holds for any $t \in \mathbb{R}$, $t \neq 0$.

Examples of functions G verifying the above conditions are:

- $G(t) = |t|^p$, $p > 1$;
- $G(t) = |t|^p + |t|^q$, $p, q > 1$;
- $G(t) = t^p(|\log(t)| + 1)$, $p > (3 + \sqrt{5})/2$;
- $G(t) = \int_0^t \left(p|s|^{p-2}s(|\log(s)| + 1) + \frac{|s|^{p-2}s}{1+|s|}\right) ds$, where $p^+ = p + 1$ and $p^- = p - 1$ (see Remark 1 in [23]).

As a final assumption, we will finally impose that g' is non-decreasing.

2.2. **Orlicz spaces.** Now, we introduce the definition of the fractional Orlicz-Sobolev spaces. For more details on this functional setting, see [14].

Definition 3. *Let $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. We define*

$$L^G(\Omega) := \{u\colon \Omega \to \mathbb{R},\ u \text{ is measurable and } \rho_{G,\Omega}(u) < \infty\}$$

and

$$W^{s,G}(\Omega) := \left\{u \in L^G(\Omega)\colon \rho_{s,G,\Omega}(u) < \infty\right\},$$

where the modulars $\rho_{G,\Omega}$ and $\rho_{s,G,\Omega}$ are defined as

$$\rho_{G,\Omega}(u) := \int_\Omega G\left(|u(x)|\right) dx,$$

$$\rho_{s,G,\Omega}(u) := \iint_{\Omega\times\Omega} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) d\mu,$$

with

$$d\mu := \frac{dxdy}{|x-y|^n}.$$

The norm associated to these spaces is the Luxemburg type norm

$$||u||_{s,G,\Omega} := ||u||_{G,\Omega} + [u]_{s,G,\Omega}$$

where

$$||u||_{G,\Omega} := \inf\left\{\lambda > 0\colon \rho_{G,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\}$$

and

$$[u]_{s,G,\Omega} := \inf\left\{\lambda > 0\colon \rho_{s,G,\Omega}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

When $\Omega = \mathbb{R}^n$, we will omit the dependence on Ω in the above quantities.

For further reference, we state the Hölder inequality.

Lemma 4. *For $u \in L^G(\Omega)$ and $v \in L^{\widetilde{G}}(\Omega)$ there holds*

$$\int_\Omega |uv| dx \le 2||u||_{G,\Omega}||v||_{\widetilde{G},\Omega}.$$

The topological dual space of $W^{s,G}(\Omega)$ is denoted by $W^{-s,\widetilde{G}}(\Omega)$. We recall that when the N-function G satisfies the $\triangle_2$ condition, then $L^G(\mathbb{R}^n)$ and $W^{s,G}(\mathbb{R}^n)$ are reflexive, separable Banach spaces (see [1, Chapter 8], [14, Proposition 2.11] and [17, Theorem 8.2]).

Also, to include boundary values, we define the space

$$W_0^{s,G}(\Omega) := \left\{u \in W^{s,G}(\mathbb{R}^n)\colon u = 0 \text{ in } \Omega^c\right\},$$

and the space of test functions

$$E := \overline{C_c^\infty(\Omega)} \subset W^{s,G}(\mathbb{R}^n)$$

where the closure is taking with respect to the norm of $\|\cdot\|_{s,G}$. Observe that $E \subset W_0^{s,G}(\Omega)$.

Next, we mention some useful lemma for N-functions, which allows comparison with power functions.

Lemma 5. [3, Lemma 2.5] *Let*

$$\xi^-(t) = \min\left\{t^{p^-}, t^{p^+}\right\}$$

and

$$\xi^+(t) = \max\left\{t^{p^-}, t^{p^+}\right\}.$$

Then,

(1) $\xi^-(||u||_{G,\Omega}) \le \rho_{G,\Omega}(u) \le \xi^+(||u||_{G,\Omega})$, *for* $u \in L^G(\Omega)$.
(2) $\xi^-([u]_{s,G}) \le \rho_{s,G}(u) \le \xi^+([u]_{s,G})$, *for* $u \in W^{s,G}(\Omega)$.

Now, we introduce a notion of comparison between N- functions.

Definition 6. *Given two N-functions A and B, we say that B is essentially larger than A, denoted by $A \ll B$, if for any $c > 0$,*

$$\lim_{t\to\infty} \frac{A(ct)}{B(t)} = 0.$$

In order for the Sobolev immersion theorem to hold, it is necessary that G verifies the following two conditions:

(G1) $\displaystyle\int_M^\infty \left(\frac{t}{G(t)}\right)^{\frac{s}{n-s}} dt = \infty$ for some M.
(G2) $\displaystyle\int_0^\delta \left(\frac{t}{G(t)}\right)^{\frac{s}{n-s}} dt < \infty$ for some $\delta > 0$.

Observe that when $G(t) = |t|^p$, then assumptions (G1) and (G2) are satisfied if $sp < n$. More examples can be seen in Remark 1 in [23].

Given G satisfying (G1) and (G2) we define its Orlicz-Sobolev conjugate as

$$G_{\frac{n}{s}}(t) = G \circ H^{-1}(t), \tag{7}$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{G(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}}.$$

Hence, we have the following embedding theorem, see [2].

Theorem 7. *Let G be an N-function satisfying* (5)*, (G1) and (G2), and let $G_{\frac{n}{s}}$ be defined in* (7)*. Then the embedding $W_0^{s,G}(\Omega) \hookrightarrow L^{G_{\frac{n}{s}}}(\Omega)$ is continuous. Moreover, the N-function $G_{\frac{n}{s}}$ is optimal in the class of Orlicz spaces. Finally, given any N-function B, the embedding $W_0^{s,G}(\Omega) \hookrightarrow L^B(\Omega)$ is compact if and only if $B \ll G_{\frac{n}{s}}$.*

2.3. **Elementary inequalities.** In this subsection we collect some elementary inequalities for N-functions G satisfying (5). We give the proofs of those which, up to our knowledge, are new in the literature.

Lemma 8. [14, Lemma 2.9] *Let G be an N-function satisfying* (5) *such that $g = G'$. Then*

$$\widetilde{G}(g(t)) \leq (p^+ - 1)G(t)$$

holds for any $t \geq 0$.

Lemma 9. *Let $a, b \in \mathbb{R}$. Then we have*

$$G\left(|a_+ - b_+|\right) \leq (a_+ - b_+)g(a - b), \tag{8}$$

where $a_+ = \max\{a, 0\}$.

Proof. Let $a, b \in \mathbb{R}$. If $a \geq 0$ and $b \leq 0$, then

$$G(|a_+ - b_+|) = G(a_+) \leq \frac{a_+}{p^-} g(a_+) \leq (a_+ - b_+)g(a - b), \tag{9}$$

where the last inequality follows from the fact that $g = G'$ is non-decreasing. The case $b_+ \geq 0$ and $a_+ = 0$ is similar.

Now, let $a_+, b_+ > 0$, i.e. $a, b > 0$. Then, if $a - b = a_+ - b_+ \geq 0$, we obtain from (9) that

$$G(|a_+ - b_+|) = G(a_+ - b_+) \leq \frac{1}{p^-}(a_+ - b_+)g(a_+ - b_+) \leq (a_+ - b_+)g(a - b).$$

If $a - b = a_+ - b_+ < 0$, and recalling that g is odd, we deduce

$$G(|a_+ - b_+|) = G(b_+ - a_+) \leq \frac{1}{p^-}(b_+ - a_+)g(b_+ - a_+) \leq (a_+ - b_+)g(a - b).$$

□

Lemma 10. *Let G be an N-function satisfying whose derivative g satisfies* (6). *Then, for all $a, b \in \mathbb{R}$ it holds*

$$g'(a - b)(a_+ - b_+)(a_+ - b_+) \leq (p^+ - 1)g(a - b)(a_+ - b_+).$$

Proof. The lemma is true if $a, b < 0$. Suppose that $a, b \geq 0$. If $a \geq b$, then, by (6),

$$g'(a-b)(a_+-b_+)(a_+-b_+) = g'(a-b)(a-b)(a_+-b_+) \leq (p^+-1)g(a-b)(a_+-b_+).$$

If $a < b$,

$$g'(a-b)(a_+-b_+)(a_+-b_+) = g'(b-a)(b-a)(b_+-a_+) \leq (p^+-1)g(b-a)(b_+-a_+).$$

On the other hand, if $a < 0$ and $b > 0$, we have

$$g'(a-b)(a_+-b_+)(a_+-b_+) = g'(a-b)(-b)(-b_+) \leq g'(b-a)(b-a)(b_+-a_+)$$

and we conclude as before. The case $a > 0$ and $b < 0$ is treated similarly. □

Lemma 11. [20, Lemma 2.1] *Assume* (6) *and that g' is non-decreasing. There exists $C > 0$ such that*

$$g(b) - g(a) \geq Cg(b - a), \tag{10}$$

for all $b \geq a$.

Remark 12. *Observe that as g is odd*

$$g(b) - g(a) \leq Cg(b - a), \qquad \textit{for all } b \leq a.$$

Lemma 13. *Assume* (6) *and that* g' *is non-decreasing. Let* a, b, c *and* d *be real numbers so that* $a - b = c - d$. *Then, there exists a constant* $C > 0$ *such that*

$$G(|a-b|) \leq C(g(c) - g(d))(a-b).$$

Proof. First, we consider the case $a > b$, using (5) and (10)

$$\begin{aligned} G(|a-b|) = G(a-b) &\leq \frac{1}{p^-} g(a-b)(a-b) = \frac{1}{p^-} g(c-d)(a-b) \\ &\leq C(g(c) - g(d))(a-b). \end{aligned}$$

The case $b > a$ follows analogously appealing to Remark 12. □

2.4. **The fractional g-Laplacian operator.** In this section we recall some elementary properties of the fractional g-Laplacian. This operator is well defined between $W^{s,G}(\mathbb{R}^n)$ and its dual space $W^{-s,\widetilde{G}}(\mathbb{R}^n)$. In fact, in [14, Theorem 6.12] the following representation formula is provided

$$\langle (-\triangle_g)^s u, v \rangle = \iint_{\mathbb{R}^{2n}} g\left(\frac{u(x) - u(y)}{|x-y|^s} \right) \frac{v(x) - v(y)}{|x-y|^s} d\mu,$$

for any $v \in W^{s,G}(\mathbb{R}^n)$.

We give some properties of the operator $(-\triangle_g)^s$ in the next result. We point out that the monotonicity of $(-\triangle_g)^s$ has been obtained in [4, Lemma 3.4] under a more restrictive assumption.

Lemma 14. $(-\triangle_g)^s \colon W_0^{s,G}(\Omega) \to W^{-s,\widetilde{G}}(\mathbb{R}^n)$ *is a monotone, continuous, and an* $(S)_+$*-operator. Moreover, it admits a continuous inverse.*

Proof. The continuity of $(-\triangle_g)^s$ follows from Hölder inequality. Observe that to prove that it is monotone and an $(S)_+$-operator, it is enough by [27, Example 27.2] to state that $(-\triangle_g)^s$ is uniformly monotone. Observe that for $u, v \in W_0^{s,G}(\Omega)$, we obtained thanks to Lemma 13 that

$$\begin{aligned} \rho_{s,G}(u-v) &= \iint_{\mathbb{R}^{2n}} G\left(\frac{(u-v)(x) - (u-v)(y)}{|x-y|^s} \right) d\mu \\ &= \iint_{\mathbb{R}^{2n}} G\left(\left| \frac{(u-v)(x) - (u-v)(y)}{|x-y|^s} \right| \right) d\mu \\ &\leq C \left\langle (-\triangle_g)^s u - (-\triangle_g)^s v, u - v \right\rangle. \end{aligned} \tag{11}$$

Thus, by Lemma 5, it follows that

$$\langle (-\triangle_g)^s u - (-\triangle_g)^s v, u - v \rangle \geq C \min\left\{ \|u-v\|_{s,G}^{p^+-1}, \|u-v\|_{s,G}^{p^--1} \right\} \|u-v\|_{s,G}.$$

We conclude that $(-\triangle_g)^s$ is uniformly monotone. To prove that it admits a continuous inverse is enough to show that is coercive, hemicontinuous and uniformly monotone (see for instance Theorem 26.A in [27]). In fact,

$$\frac{\langle (-\triangle_g)^s u, u \rangle}{[u]_{s,G}} \geq \frac{p^- \rho_{s,G}(u)}{[u]_{s,G}} \geq p^- \min\left\{ [u]_{s,G}^{p^--1}, [u]_{s,G}^{p^+-1} \right\}$$

so, taking limit as $[u]_{s,G}$ goes to infinity, we obtain that the fractional g-Laplacian is coercive. Since the real function $t \to \langle (-\triangle_g)^s (u + tv), w \rangle$ is continuous in

$[0,1]$ for any $u, v, w \in W^{s,G}(\Omega)$ we obtain that $(-\triangle_g)^s$ is hemicontinuous. This ends the proof.

□

2.5. **Notion of solution and regularity.** Let X be an ordered Banach space, its nonnegative cone is denoted by X_+.

Definition 15. *Let $u \in W^{s,G}(\mathbb{R}^n)$. Then*

(i) u is a supersolution of (1) if

$$\langle (-\triangle_g)^s u, v\rangle \geq \int_\Omega f(x,u)v\,dx, \text{ for all } v \in E_+,$$

and $u \geq 0$ in Ω^c.

(ii) u is a subsolution of (1) if

$$\langle (-\triangle_g)^s u, v\rangle \leq \int_\Omega f(x,u)v\,dx, \text{ for all } v \in E_+,$$

and $u \leq 0$ in Ω^c.

Moreover , if $\underline{u}$ is a subsolution, $\overline{u}$ is a supersolution, and $\underline{u} \leq \overline{u}$ in Ω, then the pair $(\underline{u}, \overline{u}) \in W^{s,G}(\mathbb{R}^n) \times W^{s,G}(\mathbb{R}^n)$ is called a sub-supersolution pair of (1).

Definition 16. *We say that $u \in W_0^{s,G}(\Omega)$ is a weak solution of (1) if*

$$\langle (-\triangle_g)^s u, v\rangle = \int_\Omega f(x,u)v\,dx, \text{ for all } v \in E.$$

Finally, we mention some results concerning the regularity of solutions to the g-Laplace equation. We refer the reader to [15, Proposition 4.7], [15, Theorem 1.1] and [16, Theorem 3].

Theorem 17. *Let $s \in (0,1)$ and let G be an N-function satisfying (5) with $sp^+ < n$. Moreover, assume that*

$$\int_\Omega G(u(x))\,dx > 0, \tag{12}$$

where $u \in W_0^{s,G}(\Omega)$ is a weak solution of the problem

$$\begin{cases} (-\Delta_g)^s u = b(u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $b = B'$, with B an N-function satisfying

$$\eta^- \leq \frac{tb(t)}{B(t)} \leq \eta^+,$$

and $B \ll G_{\frac{n}{s}}$. Then, there exists a constant $C = C(s, n, p^\pm, \eta^\pm) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

If G additionally satisfies (6), with $p^- > 2$ and g is a convex function in $(0,\infty)$, then there exists $\alpha \in (0,1)$ such that

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C.$$

3. SOLUTIONS IN A SUB-SUPERSOLUTION INTERVAL

In this section we extend the results obtained in [11] to the Orlicz fractional setting. More precisely, we focus in proving the existence of solutions for (1). We assume that the function f satisfies the following hypothesis:

($\mathbf{H}_0$) $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for a.e $x\in\Omega$ and all $t\in\mathbb{R}$

$$|f(x,t)|\le c_0(1+|b(t)|),$$

where $c_0>0$ and $b(t)=B'(t)$, with B an N-function satisfying (5) and $G\ll B\ll G_{\frac{n}{s}}$.

We point out that this kind of hypothesis has been used before in the literature, see for example [3].

Now, we are in position to prove our first lemma.

Lemma 18. *Suppose that f satisfies ($\boldsymbol{H}_0$). Let $u_1,u_2\in W^{s,G}(\mathbb{R}^n)$:*

(i) if u_1,u_2 are supersolutions of (1), then $u^*=\min\{u_1,u_2\}$ is also a supersolution;

(ii) if u_1,u_2 are subsolutions of (1), then $u_*=\max\{u_1,u_2\}$ is a subsolution as well.

Proof. We prove (i), the proof of (ii) is analogous. We have for $i=1,2$ and for all $v\in E_+$ that

$$\left\{\begin{array}{c}\displaystyle\iint_{\mathbb{R}^{2n}} g\left(\frac{u_i(x)-u_i(y)}{|x-y|^s}\right)\frac{v(x)-v(y)}{|x-y|^s}d\mu\ge\int_\Omega f(x,u_i)v dx,\\ u_i\ge 0 \text{ in } \Omega^c.\end{array}\right. \tag{13}$$

Set $u^*=\min\{u_1,u_2\}\in W^{s,G}(\mathbb{R}^n)$, then $u^*\ge0$ in Ω^c. Moreover, we consider the following sets

$$A_1=\{x\in\mathbb{R}^n\colon u_1(x)<u_2(x)\},\quad A_2=A_1^c.$$

Fix $\varepsilon>0$, and define for all $t\in\mathbb{R}$ the truncation function

$$\tau_\varepsilon(t)=\begin{cases}0 & if\ t\le0,\\ \dfrac{t}{\varepsilon} & if\ 0<t<\varepsilon,\\ 1 & if\ t\ge\varepsilon.\end{cases}$$

The mapping $\tau_\varepsilon\colon\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, non-decreasing, $0\le\tau_\varepsilon(t)\le1$ for all $t\in\mathbb{R}$, and

$$\tau_\varepsilon(u_2-u_1)\to\mathcal{X}_{A_1},\qquad 1-\tau_\varepsilon(u_2-u_1)\to\mathcal{X}_{A_2} \tag{14}$$

a.e in $\mathbb{R}^n$ as $\varepsilon\to0^+$. For simplicity, we denote

$$\tau_\varepsilon:=\tau_\varepsilon(u_2-u_1)$$

and

$$1-\tau_\varepsilon:=1-\tau_\varepsilon(u_2-u_1).$$

Let now $\varphi \in C_c^\infty(\Omega)_+$. Using $\tau_\varepsilon\varphi$ and $1-\tau_\varepsilon\varphi$ as test functions in (13), we obtain

$$
\begin{aligned}
I :=& \iint_{\mathbb{R}^{2n}} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\tau_\varepsilon(x)\varphi(x)-\tau_\varepsilon(y)\varphi(y)}{|x-y|^s}d\mu \\
&+ \iint_{\mathbb{R}^{2n}} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{(1-\tau_\varepsilon)(x)\varphi(x)-(1-\tau_\varepsilon)(y)\varphi(y)}{|x-y|^s}d\mu \\
\geq& \int_\Omega f(x,u_1)\tau_\varepsilon\varphi dx + \int_\Omega f(x,u_2)(1-\tau_\varepsilon)\varphi dx := II.
\end{aligned} \tag{15}
$$

We get, for A_1 and A_2,

$$
\begin{aligned}
I =& \iint_{A_1\times A_1} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)]d\mu && (A)\\
&+ \iint_{A_1\times A_1} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)-\tau_\varepsilon(y)]d\mu && (B)\\
&+ \iint_{A_1\times A_2} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}[\tau_\varepsilon(x)]d\mu && (C)\\
&- \iint_{A_2\times A_1} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}[\tau_\varepsilon(y)]d\mu && (D)\\
&+ \iint_{A_1\times A_1} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}[1-\tau_\varepsilon(x)]d\mu && (E)\\
&- \iint_{A_1\times A_1} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)-\tau_\varepsilon(y)]d\mu && (B)\\
&+ \iint_{A_1\times A_2} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}[1-\tau_\varepsilon(x)]d\mu && (F)\\
&- \iint_{A_1\times A_2} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)]d\mu && (C)\\
&+ \iint_{A_2\times A_1} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}[\tau_\varepsilon(y)]d\mu && (D)\\
&+ \iint_{A_2\times A_1} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}[1-\tau_\varepsilon(y)]d\mu && (G)\\
&+ \iint_{A_2\times A_2} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}d\mu && (H).
\end{aligned}
$$

Observe first that $(E),(F),(G)\to 0$ as $\varepsilon\to 0^+$ by (14). Also, getting together the integrals with equal letters, we have

$$
\begin{aligned}
I &= \iint_{A_1\times A_1} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)]d\mu\\
&+\iint_{A_1\times A_1}\left[g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)-g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\right]\frac{\varphi(y)}{|x-y|^s}[\tau_\varepsilon(x)-\tau_\varepsilon(y)]d\mu\\
&+\iint_{A_1\times A_2}\left[g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}-g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}\right]\tau_\varepsilon(x)d\mu\\
&+\iint_{A_2\times A_1}\left[g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)}{|x-y|^s}-g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(y)}{|x-y|^s}\right]\tau_\varepsilon(y)d\mu\\
&+\iint_{A_2\times A_2} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}d\mu\\
&+\mathbf{o}(1)\\
&=(A)+(B)+(C)+(D)+(H).
\end{aligned}
$$

We have that (B) is negative, since g is non-decreasing and since for all $x,y\in A_1$

$$u_1(x)-u_1(y)\geq u_2(x)-u_2(y)$$

if and only if

$$\tau_\varepsilon(y)=\tau_\varepsilon(u_2-u_1)(y)\geq\tau_\varepsilon(u_2-u_1)(x)=\tau_\varepsilon(x).$$

On the other hand, for all $x\in A_1$, $y\in A_2$, there holds

$$u_1(x)-u_1(y)\leq u_1(x)-u_2(y)\leq u_2(x)-u_2(y),$$

and thus, using again that g is non-decreasing, we get

$$g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\leq g\left(\frac{u_1(x)-u_2(y)}{|x-y|^s}\right)\leq g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right).$$

Then, (C) is bounded from above by

$$\iint_{A_1\times A_2}\left[g\left(\frac{u_1(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\right]\tau_\varepsilon(x)d\mu.$$

Analogously, for all $x\in A_2$, $y\in A_1$ we can bound (D) by

$$\iint_{A_2\times A_1}\left[g\left(\frac{u_2(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\right]\tau_\varepsilon(y)d\mu.$$

So we have

$$
\begin{aligned}
I \;\; &\leq \iint_{A_1\times A_1} g\left(\frac{u_1(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\tau_\varepsilon(x)d\mu \\
&+ \iint_{A_1\times A_2} g\left(\frac{u_1(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\tau_\varepsilon(x)d\mu \\
&+ \iint_{A_2\times A_1} g\left(\frac{u_2(x)-u_1(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\tau_\varepsilon(y)d\mu \\
&+ \iint_{A_2\times A_2} g\left(\frac{u_2(x)-u_2(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}d\mu \; + \mathbf{o}(1) \\
&\to \iint_{\mathbb{R}^{2n}} g\left(\frac{u^*(x)-u^*(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}d\mu, \quad \text{as } \varepsilon\to 0.
\end{aligned}
$$

Now in order to estimate II, we use the Dominated Convergence Theorem. Observe that the mayorant is obtained thanks to $(\mathbf{H}_0)$ and the definition of τ_ε. In fact,

$$
|f(\cdot,u_1)\tau_\varepsilon\varphi| \leq c_0(1+|b(u_1)|)\varphi,
$$
$$
|f(\cdot,u_2)(1-\tau_\varepsilon)\varphi| \leq c_0(1+|b(u_2)|)\varphi.
$$

So, passing to the limit as $\varepsilon\to 0^+$,

$$
II \to \int_\Omega f(x,u^*)\varphi dx. \tag{16}
$$

Hence, taking $\varepsilon\to 0^+$ in (15), we get for all $\varphi\in C_c^\infty(\Omega)_+$ that

$$
\iint_{\mathbb{R}^{2n}} g\left(\frac{u^*(x)-u^*(y)}{|x-y|^s}\right)\frac{\varphi(x)-\varphi(y)}{|x-y|^s}d\mu \geq \int_\Omega f(x,u^*)\varphi dx.
$$

Finally by the definition of E, we obtain that the previous inequality holds for all test functions in E_+. Thus, u^* is a supersolution of (1), as we want to prove. □

Our next lemma proves that $S(\underline{u},\overline{u})$ is not empty.

Lemma 19. *Let* $(\underline{u},\overline{u})$ *be a sub-supersolution pair of* (1) *with* f *such that verifies* $\boldsymbol{H}_0$. *Then, there exists* $u\in S\,(\underline{u},\overline{u})$.

Proof. For simplicity, we let $A=(-\triangle_g)^s\colon W_0^{s,G}(\Omega)\to W^{-s,\widetilde{G}}(\Omega)$. By Lemma 14, A is monotone and hemicontinuous. Thanks to [22, Lemma 2.98 (i)] we know that also A is pseudomonotone.

We consider a truncation of f defined as

$$
\widetilde{f}(x,t)=\begin{cases} f(x,\underline{u}(x)) & if\; t\leq \underline{u}(x),\\ f(x,t) & if\; \underline{u}(x)<t<\underline{u}(x),\\ f(x,\overline{u}(x)) & if\; t\geq \overline{u}(x),\end{cases}
$$

for all $(x,t)\in\Omega\times\mathbb{R}$. Observe that $\widetilde{f}$ does not satisfy $\mathbf{H}_0$ in general, however $\widetilde{f}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which satisfies the following bound for a.e. x and all $t\in\mathbb{R}$,

$$
|\widetilde{f}(x,t)|\leq c_0(1+|b(\overline{u}(x))|+|b(\underline{u}(x))|)\in L^{\widetilde{B}}(\Omega). \tag{17}
$$

We define $D\colon W_0^{s,G}(\Omega)\to W^{-s,\widetilde{G}}(\Omega)$ as

$$\langle D(u),v\rangle=-\int_\Omega \widetilde{f}(x,u)v\,dx,$$

for all $u,v\in W_0^{s,G}(\Omega)$. Observe that D is well defined since by (17) and $G\ll B$, we obtain

$$\begin{aligned}\left|-\int_\Omega \widetilde{f}(x,u)v\,dx\right| &\le \int_\Omega |\widetilde{f}(x,u)||v|\,dx\\ &\le 2||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{B,\Omega}\\ &\le C||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{s,G,\Omega}\\ &<\infty.\end{aligned}$$

Moreover, D is linear and continuous.

We now prove that D is strongly continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n\rightharpoonup u$ in $W_0^{s,G}(\Omega)$, and take any subsequence of $\{D(u_n)\}_{n\in\mathbb{N}}$ that we still denote by $\{D(u_n)\}_{n\in\mathbb{N}}$. Passing to a further subsequence if necessary, we have $u_n\to u$ in $L^B(\Omega)$, $u_n(x)\to u(x)$ and $|u_n(x)|\le h(x)$ for a.e. $x\in\Omega$, for some $h\in L^B(\Omega)$. Therefore, for all $n\in\mathbb{N}$, by $\mathbf{H}_0$ and Lemma 8 we have for a.e. $x\in\Omega$

$$\begin{aligned}|\widetilde{f}(x,u_n)-\widetilde{f}(x,u)| &\le |\widetilde{f}(x,u_n)|+|\widetilde{f}(x,u)|\\ &=2c_0+c_0|b(\overline{u}_n)|+c_0|b(\underline{u}_n)|+c_0|b(\overline{u})|+c_0|b(\underline{u})|\in L^{\widetilde{B}}(\Omega).\end{aligned}$$

Observe that by continuity of $f(x,\cdot)$ we have $\widetilde{f}(x,u_n)\to\widetilde{f}(x,u)$ a.e. Moreover, for all $v\in W_0^{s,G}(\Omega)$,

$$\begin{aligned}|\langle D(u_n)-D(u),v\rangle| &\le \int_\Omega |\widetilde{f}(x,u_n)-\widetilde{f}(x,u)||v|dx\\ &\le 2||\widetilde{f}(x,u_n)-\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{B,\Omega}\\ &\le C||\widetilde{f}(x,u_n)-\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{s,G,\Omega}.\end{aligned}\tag{18}$$

By dominated convergence Theorem, the right-hand side of (18) tends to 0 uniformly for any v with $\|v\|_{s,G,\Omega}\le 1$. Hence $D(u_n)\to D(u)$ in $W^{-s,\widetilde{G}}(\Omega)$. Therefore, any subsequence of $\{D(u_n)\}_{n\in\mathbb{N}}$ has a further subsequence that converges to $D(u)$. Hence, the whole sequence $\{D(u_n)\}_{n\in\mathbb{N}}$ converges to $D(u)$ and D is strongly continuous. Thanks to [22, Lemma 2.98 (ii)], D is pseudomonotone. Moreover, $A+D$ is pseudomonotone by [27, Proposition 27.7].

Now we prove that $A+D$ sends bounded sets into bounded set. We already stated that A is bounded. Regarding D, observe that

$$\begin{aligned}
||D(u)||_{-s,\widetilde{G}} &= \sup_{||v||_{s,G,\Omega}\leq 1} \langle D(u), v\rangle \\
&= \sup_{||v||_{s,G,\Omega}\leq 1} -\int_\Omega \widetilde{f}(x,u)v dx \\
&\leq \sup_{||v||_{s,G,\Omega}\leq 1} 2||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{B,\Omega} \\
&\leq C \sup_{||v||_{s,G,\Omega}\leq 1} ||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||v||_{s,G,\Omega} \\
&\leq C\|c_0(1+|b(\overline{u})|+|b(\underline{u})|)\|_{\widetilde{B},\Omega}.
\end{aligned}$$

Thus, D sends bounded sets into bounded sets.

It remains to check that $A+D$ is coercive. In fact, for all $u\in W_0^{s,G}(\Omega)$ with $||u||_{s,G}\geq 1$ we have

$$\begin{aligned}
\frac{\langle A(u)+D(u),u\rangle}{||u||_{s,G}} &= \frac{\langle A(u),u\rangle}{||u||_{s,G}} + \frac{\langle D(u),u\rangle}{||u||_{s,G}} \\
&\geq \frac{p^-\rho_{s,G}(u)}{||u||_{s,G}} - \frac{1}{||u||_{s,G}}\int_\Omega \widetilde{f}(x,u)u\, dx \\
&\geq \frac{p^-\xi^-([u]_{s,G})}{||u||_{s,G}} - \frac{1}{||u||_{s,G}}\int_\Omega |\widetilde{f}(x,u)||u|\, dx \\
&\geq p^-\,[u]_{s,G}^{p^--1} - \frac{2}{||u||_{s,G}}||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||u||_{B,\Omega} \\
&\geq p^-\,[u]_{s,G}^{p^--1} - \frac{C}{||u||_{s,G}}||\widetilde{f}(x,u)||_{\widetilde{B},\Omega}||u||_{s,G} \\
&\geq p^-\,[u]_{s,G}^{p^--1} - C.
\end{aligned}$$

Observe that this goes to infinity when $[u]_{s,G}\to\infty$, as we want to prove.

Appealing to [22, Theorem 2.99], the equation

$$A(u)+D(u)=0 \text{ in } W^{-s,\widetilde{G}}(\Omega) \tag{19}$$

has a solution $u\in W_0^{s,G}(\Omega)$. Next, we prove that in Ω

$$\underline{u}\leq u\leq \overline{u}. \tag{20}$$

Observe that (20) holds in Ω^c. Now we use $(u-\overline{u})_+\in W_0^{s,G}(\Omega)_+$ as a test function in (19). Then

$$\langle A(u)+D(u),(u-\overline{u})_+\rangle = \langle A(u),(u-\overline{u})_+\rangle + \langle D(u),(u-\overline{u})_+\rangle = 0.$$

Thus,

$$\langle (-\triangle_g)^s u,(u-\overline{u})_+\rangle = \int_\Omega \widetilde{f}(x,u)(u-\overline{u})_+\, dx.$$

Observe now that in the support of $(u-\overline{u})_+$ we have $\widetilde{f}(x,u)=f(x,\overline{u})$. Hence,

$$\begin{aligned}&\langle(-\triangle_g)^s u,(u-\overline{u})_+\rangle=\int_\Omega f(x,\overline{u})(u-\overline{u})_+\,dx\\&\le\iint_{\mathbb{R}^{2n}} g\left(\frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^s}\right)\frac{(u-\overline{u})_+(x)-(u-\overline{u})_+(y)}{|x-y|^s}d\mu\\&=\langle(-\triangle_g)^s\overline{u},(u-\overline{u})_+\rangle,\end{aligned}$$

where we also used that $\overline{u}$ is a supersolution of (1), so

$$\langle(-\triangle_g)^s u-(-\triangle_g)^s\overline{u},(u-\overline{u})_+\rangle\le 0. \tag{21}$$

In the next calculations, we first use (8) with

$$a=\frac{(u-\overline{u})(x)}{|x-y|^s}$$

and

$$b=\frac{(u-\overline{u})(y)}{|x-y|^s}.$$

Also, by (10) and the fact that $u(x)-u(y)\le\overline{u}(x)-\overline{u}(y)$ if and only if $(u-\overline{u})_+(x)\le(u-\overline{u})_+(y)$, we have

$$\begin{aligned}&\rho_{s,G}\left((u-\overline{u})_+\right)=\iint_{\mathbb{R}^{2n}} G\left(\frac{|(u-\overline{u})_+(x)-(u-\overline{u})_+(y)|}{|x-y|^s}\right)d\mu\\&\le\iint_{\mathbb{R}^{2n}} g\left(\frac{(u-\overline{u})(x)-(u-\overline{u})(y)}{|x-y|^s}\right)\frac{(u-\overline{u})_+(x)-(u-\overline{u})_+(y)}{|x-y|^s}d\mu\\&\le\frac{1}{C}\iint_{\mathbb{R}^{2n}}\left[g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)-g\left(\frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^s}\right)\right]\frac{(u-\overline{u})_+(x)-(u-\overline{u})_+(y)}{|x-y|^s}d\mu\\&=\frac{1}{C}\iint_{\mathbb{R}^{2n}} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{(u-\overline{u})_+(x)-(u-\overline{u})_+(y)}{|x-y|^s}d\mu\\&-\frac{1}{C}\iint_{\mathbb{R}^{2n}} g\left(\frac{\overline{u}(x)-\overline{u}(y)}{|x-y|^s}\right)\frac{(u-\overline{u})_+(x)-(u-\overline{u})_+(y)}{|x-y|^s}d\mu\\&=C'\left[\langle(-\triangle_g)^s u-(-\triangle_g)^s\overline{u},(u-\overline{u})_+\rangle\right].\end{aligned}$$

Consequently, by (21), it follows $(u-\overline{u})_+=0$. Analogously we prove $u\ge\underline{u}$, so the inequality (20) holds.

Finally, using (20) in (19) we see that $u\in W_0^{s,G}(\Omega)$ solves (1) in the sense of Definition 16 (since $E\subset W^{s,G}(\mathbb{R}^n)$) and $u\in S(\underline{u},\overline{u})$ as we want to prove. □

We end the section giving a result related to extremal elements in $S(\underline{u},\overline{u})$. For that, we will need two preliminary results. The proofs below are similar to those from [11]. However, we write them for convenience of the reader.

We recall the following definition.

Definition 20. *We say that a partially ordered set $(S,\le)$ is* downward directed *(resp.* upward directed*) if for all $u_1,u_2\in S$ there exists $u_3\in S$ such that $u_3\le u_1,u_2$ (resp. $u_3\ge u_1,u_2$).*

Also, we say that S is directed *if it is both downward and upward directed.*

Lemma 21. *Let $(\underline{u},\overline{u})$ be a sub-supersolution pair of* (1) *with f such that verifies $\boldsymbol{H}_0$. Then, $S(\underline{u},\overline{u})$ is directed.*

Proof. We prove that $S(\underline{u},\overline{u})$ is downward directed. Let $u_1, u_2 \in S(\underline{u},\overline{u})$, then in particular u_1, u_2 are supersolutions of (1). Set $u^* = \min\{u_1, u_2\} \in W_0^{s,G}(\Omega)$, then by Lemma 18 u^* is a supersolution of (1) and $\underline{u} \le u^*$. By Lemma 19 there exists $u_3 \in S(\underline{u}, u^*)$, in particular $u_3 \in S(\underline{u},\overline{u})$ and $u_3 \le u^*$.

In the same way we see that $S(\underline{u},\overline{u})$ is upward directed. And then $S(\underline{u},\overline{u})$ is directed. □

Lemma 22. *Let* $(\underline{u},\overline{u})$ *be a sub-supersolution pair of* (1) *with* f *such that verifies* $\boldsymbol{H}_0$. *Then* $S(\underline{u},\overline{u})$ *is compact in* $W_0^{s,G}(\Omega)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $S(\underline{u},\overline{u})$, then for all $n \in \mathbb{N}$ and $v \in W_0^{s,G}(\Omega)$

$$\langle(-\triangle_g)^s u_n, v\rangle = \int_\Omega f(x,u_n)v\,dx \tag{22}$$

and $\underline{u} \le u_n \le \overline{u}$. Using $u_n \in W_0^{s,G}(\Omega)$ as a test function in (22), we have by $\mathbf{H}_0$

$$\begin{aligned}
\langle(-\triangle_g)^s u_n, u_n\rangle &\le \int_\Omega |f(x,u_n)||u_n|\,dx \\
&\le c_0\left[\int_\Omega |u_n|dx + \int_\Omega |b(u_n)||u_n|\,dx\right] \\
&\le c_0\left[\int_\Omega |u_n|dx + p_B^+\int_\Omega B(|u_n|)\,dx\right] \\
&\le c_0\left[\|\underline{u}\|_{1,\Omega} + \|\overline{u}\|_{1,\Omega} + p_B^+\frac{C}{2}\rho_{B,\Omega}(\underline{u}) + p_B^+\frac{C}{2}\rho_{B,\Omega}(\overline{u})\right] \le C.
\end{aligned}$$

So $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{s,G}(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup u$ in $W_0^{s,G}(\Omega)$, $u_n(x) \to u(x)$ and $|u_n| \le h$ for a.e $x \in \Omega$ and $n \in \mathbb{N}$, with $h \in L^G(\Omega)$. Moreover,

$$|f(x,u_n)(u_n-u)| \le 2c_0\,(1+|b(h)|)\,(|\underline{u}|+|\overline{u}|) \in L^1(\Omega).$$

Now, using $u_n - u \in W_0^{s,G}(\Omega)$ as a test function in (22), we get

$$\langle(-\triangle_g)^s u_n, u_n-u\rangle = \int_\Omega f(x,u_n)(u_n-u)\,dx. \tag{23}$$

Using Dominated convergence Theorem we have that (23) tends to 0 as $n \to \infty$. By Lemma 14 we have $u_n \to u$ in $W_0^{s,G}(\Omega)$. Finally, taking limit in (22) we obtain that $u \in S(\underline{u},\overline{u})$. □

As an application of the previous results, we obtain proceeding exactly as in [11, Theorem 3.5] that $S(\underline{u},\overline{u})$ contains extremal elements with respect to the pointwise ordering. Also, observe that we apply Theorem 17 to have $C^\alpha(\overline{\Omega})$ regularity of the solutions.

Theorem 23. *Assume that* $f(u) = b(u)$, $b(0) = 0$, *where* $b = B'$, *with* B *an N-function so that* $G \ll B \ll G_{\frac{n}{s}}$ *and it satisfies* (5). *Moreover, assume that* G *fulfils* (12) *and* (6), *with* $g = G'$ *convex in* $(0,\infty)$ *and* $p^- > 2$. *Let* $(\underline{u},\overline{u})$ *be a sub-supersolution pair of* (1). *Then* $S(\underline{u},\overline{u})$ *contains a smallest and a biggest element.*

4. MULTIPLE SOLUTIONS

In this section, we follow the approach of [26] to prove the existence of three non-trivial different solutions of (1), two of constant sign and one nodal.

Throughout this section, we assume that the N-function G satisfies (6).

We let

$$F(x,u) := \int_0^u f(x,\tau)\,d\tau,$$

where f satisfies $(\mathbf{H}_0)$ and the following hypothesis:

$(\mathbf{H}_1)$ $f(x,0)=0$ for a.e. x. Also, f is continuously differentiable in u.

$(\mathbf{H}_2)$ There are an N-function B with $G \ll B \ll G_{\frac{n}{s}}$ and $p^+ < p_B^-$, and constants $c_i > 0$, for $i = 1,2,3,4$, with

$$c_2 > p^+, \quad c_3 < \frac{1}{p^+-1},$$

such that for all $u \in L^B(\Omega)$ there holds

$$c_1\rho_{B,\Omega}(u) \le c_2\int_\Omega F(x,u)\,dx \le \int_\Omega f(x,u)u\,dx \le c_3\int_\Omega f_u(x,u)u^2\,dx \le c_4\rho_{B,\Omega}(u).$$

Remark 24. *Observe that* $(\mathbf{H}_1)$ *implies that the trivial function* $u \equiv 0$ *is a solution of* (1).

Remark 25. *Assume that* $f(u) = b(u)$*,* $b(0)=0$*, where* $b = B'$*,* B *an N-function so that* $G \ll B \ll G_{\frac{n}{s}}$*,*

$$p_B^- - 1 \le \frac{tb'(t)}{b(t)} \le p_B^+ - 1,$$

and $p^+ < p_B^-$*. Then* f *satisfies* $(\mathbf{H}_0), (\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$*. Indeed,* $(\mathbf{H}_0), (\mathbf{H}_1)$ *are clearly satisfied and for* $(\mathbf{H}_2)$ *observe that*

$$\begin{aligned}p_B^-\rho_{B,\Omega}(u) &= p_B^-\int_\Omega B(u(x))\,dx \le \int_\Omega b(|u(x)|)|u(x)|\,dx \\ &= \int_\Omega b(u(x))u(x)\,dx \le \frac{1}{p_B^- - 1}\int_\Omega b'(|u(x)|)u(x)^2\,dx \le \frac{p_B^+(p_B^+-1)}{p_B^- - 1}\rho_{B,\Omega}(u).\end{aligned}$$

Hence, taking $c_1 = c_2 = p_B^-$*,* $c_3 = 1/(p_B^- - 1)$*, and* $c_4 = p_B^+(p_B^+-1)/(p_B^- - 1)$ *we get* $(\mathbf{H}_2)$*.*

More generally, suppose that $f(x,\cdot)$ *is odd,* $f(x,u) \ge 0$ *for* $u \ge 0$*, and the following pointwise relations hold:*

$$0 < q^- - 1 \le \frac{uf_u(x,u)}{f(x,u)} \le q^+ - 1, \quad p^+ + 1 < q^- \le q^+, \tag{24}$$

and

$$b(u) \le q^- f(x,u), \ f_u(x,u) \le cb'(u), \tag{25}$$

for some $c > 0$ *and all* x *and* u*. Then* $(\mathbf{H}_2)$ *holds. Indeed, we first have by* (25)*,*

(26)
$$\begin{aligned}\left(1-\frac{1}{q^-}\right)\rho_{B,\Omega}(u) &= \left(1-\frac{1}{q^-}\right)\int_\Omega\int_0^{|u(x)|} b(t)\,dt\,dx\\ &\leq (q^- -1)\int_\Omega F(x,|u(x)|)\,dx = (q^- -1)\int_\Omega F(x,u(x))\,dx,\end{aligned}$$

where we have used that f is odd in the last equality. This proves the first integral inequality in $(\mathbf{H}_2)$ *with $c_1 = 1-\frac{1}{q^-}$ and $c_2 = q^- - 1$. Next,*

$$\begin{aligned}(q^- -1)\int_\Omega F(x,|u(x)|)\,dx &= (q^- -1)\int_\Omega\int_0^{|u(x)|} f(x,t)\,dt\,dx\\ &\leq \int_\Omega\int_0^{|u(x)|} tf_t(x,t)\,dt\,dx\\ &= \int_\Omega\left(tf(x,t)\Big|_0^{|u(x)|} - \int_0^{|u(x)|} f(x,t)\right)dx\\ &\leq \int_\Omega |u|f(x,|u|)\,dx\\ &= \int_\Omega uf(x,u)\,dx.\end{aligned}$$

Finally, observe that by (24) *and* (25)*,*

$$\int_\Omega uf(x,u)\,dx \leq \frac{1}{q^- -1}\int_\Omega u^2 f_u(x,u)\,dx \leq \frac{cp_B^+(p_B^+-1)}{q^- -1}\rho_{B,\Omega}(u),$$

which ends the proof taking $c_3 = 1/(q^- -1)$ and $c_4 = cp_B^+(p_B^+-1)/(q^- -1)$.

Observe that weak solutions of (1) are critical points of the functional $\Phi : W_0^{s,G}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) := \iint_{\mathbb{R}^{2n}} G\left(\left|\frac{u(x)-u(y)}{|x-y|^s}\right|\right)\,d\mu - \mathcal{F}(u),$$

where

$$\mathcal{F}(u) := \int_\Omega F(x,u)\,dx.$$

Define the following subsets of $W_0^{s,G}(\Omega)$:

$$M_1 := \left\{u \in W_0^{s,G}(\Omega) : \int_\Omega u_+ dx > 0 \text{ and } \langle(-\triangle_g)^s u, u_+\rangle = \left\langle \mathcal{F}'(u), u_+\right\rangle\right\},$$

$$M_2 := \left\{u \in W_0^{s,G}(\Omega) : \int_\Omega u_- dx > 0 \text{ and } \langle(-\triangle_g)^s u, u_-\rangle = \left\langle \mathcal{F}'(u), u_-\right\rangle\right\},$$

where $u_- = \max\{-u, 0\}$ and

$$M_3 := M_1 \cap M_2.$$

Finally, we define

$$K_1 := \{u \in M_1 : u \geq 0\},$$
$$K_2 := \{u \in M_2 : u \leq 0\}$$

and

$$K_3 := M_3.$$

We start with the next lemma to show that these sets are not empty. We borrow some calculation from [5] and write the proof in details for completeness.

Lemma 26. *The sets M_1, M_2 and M_3 are non empty.*

Proof. Let

$$\varphi_1(w) = \langle (-\triangle_g)^s w, w\rangle - \int_\Omega f(x,w)w\,dx.$$

Take any $w_0 \in W_0^{s,G}(\Omega)_+$, with $w_{0+} \neq 0$ in a set of positive measure. Observe that if $0 < t < 1$, by $(\mathbf{H}_2)$,

$$\begin{aligned}
\varphi_1(tw_0) &= \langle (-\triangle_g)^s tw_0, tw_0\rangle - \int_\Omega f(x,tw_0)tw_0\,dx \\
&\geq \iint_{\mathbb{R}^{2n}} g\left(\frac{tw_0(x)-tw_0(y)}{|x-y|^s}\right)\frac{tw_0(x)-tw_0(y)}{|x-y|^s}\,d\mu - c_4\rho_{B,\Omega}(tw_0) \\
&\geq t^{p^+}\iint_{\mathbb{R}^{2n}} g\left(\frac{w_0(x)-w_0(y)}{|x-y|^s}\right)\frac{w_0(x)-w_0(y)}{|x-y|^s}\,d\mu - c_4\rho_{B,\Omega}(tw_0) \\
&\geq t^{p^+}\langle (-\triangle_g)^s w_0, w_0\rangle - c_4 t^{p_B^-}\rho_{B,\Omega}(w_0) \\
&\geq t^{p^+}A_1 - c_4 t^{p_B^-}A_2,
\end{aligned}$$

where $A_1 = \langle (-\triangle_g)^s w_0, w_0\rangle \geq C\rho_{s,G}(w_0) \neq 0$ and $A_2 = \rho_{B,\Omega}(w_0)$. As $p^+ < p_B^-$ it follows that $\varphi_1(tw_0) \geq 0$ for t small enough. On the other hand, if $t \geq 1$, analogously

$$\varphi_1(tw_0) \leq t^{p^-}A_1 - c_1 t^{p_B^+}A_2.$$

As $p^- < p^+ < p_B^- < p_B^+$ it follows that $\varphi_1(tw_0) \leq 0$ for t big enough. Finally, by Bolzano's theorem there exists $t_0 > 0$ such that $t_0 w_0 \in M_1$. Analogously if $w_0 \leq 0$ with $w_0 < 0$ in a set of positive measure, there exists $t_1 > 0$ such that $t_1 w_0 \in M_2$. Finally, we prove that $M_3 \neq \emptyset$. Let $w \in W_0^{s,G}(\Omega)$, with $w_+, w_- \neq 0$. For $s, t > 0$, define

$$\varphi(t,s) := (\varphi_1(t,s), \varphi_2(t,s)),$$

where

$$\varphi_1(t,s) := \langle (-\triangle_g)^s(tw_+ - sw_-), tw_+\rangle - \int_\Omega f(x, tw_+ - sw_-)tw_+\,dx$$

and

$$\varphi_2(t,s) := \langle (-\triangle_g)^s(tw_+ - sw_-), s(-w_-)\rangle - \int_\Omega f(x, tw_+ - sw_-)s(-w_-)\,dx.$$

Then, for $t \in [0,1]$,

(27)

$$\begin{aligned}
\varphi_1(t,t) = &\iint_{\mathbb{R}^{2n}} g\left(\frac{(tw_+ - tw_-)(x) - (tw_+ - tw_-)(y)}{|x-y|^s}\right) \frac{tw_+(x) - tw_+(y)}{|x-y|^s}\, d\mu \\
&- \int_\Omega f(x, tw_+ - tw_-) tw_+ \, dx \\
&\geq t^{p^+} \iint_{\mathbb{R}^{2n}} g\left(\frac{(w_+ - w_-)(x) - (w_+ - w_-)(y)}{|x-y|^s}\right) \frac{w_+(x) - w_+(y)}{|x-y|^s}\, d\mu \\
&- \int_\Omega f(x, tw_+) tw_+ \, dx \\
&\geq t^{p^+} A_1 - c_4 t^{p_B^-} A_2,
\end{aligned}$$

where $A_1 \geq \rho_{s,G}(w_+) \neq 0$ (by 8) and $A_2 = \rho_{B,\Omega}(w_+)$. Thus, for t small we get $\varphi_1(t,t) > 0$. Similarly, it holds $\varphi_2(t,t) > 0$. On the other hand, it also holds that for t large that $\varphi_1(t,t) < 0$ and $\varphi_2(t,t) < 0$. Next, we show that $\varphi_1(t,s)$ is increasing in s for fixed t. Let $s_1 \leq s_2$ and consider as in [5] the sets

$$D_1 = \{x \in \mathbb{R}^n : w(x) \geq 0\}, \quad D_2 := D_1^c.$$

Then,

$$\begin{aligned}
&\varphi_1(t,s_2) - \varphi_1(t,s_1) \\
&\quad = \langle (-\triangle_g)^s (tw_+ - s_2 w_-), tw_+ \rangle - \int_\Omega f(x, tw_+ - s_2 w_-) tw_+ \, dx \\
&\quad - \langle (-\triangle_g)^s (tw_+ - s_1 w_-), tw_+ \rangle + \int_\Omega f(x, tw_+ - s_1 w_-) tw_+ \, dx \\
&\quad\quad = \langle (-\triangle_g)^s (tw_+ - s_2 w_-), tw_+ \rangle - \langle (-\triangle_g)^s (tw_+ - s_1 w_-), tw_+ \rangle \\
&\quad = \int_{D_2}\int_{D_1} g\left(\frac{tw_+(x) + s_2 w_-(y)}{|x-y|^s}\right) \frac{tw_+(x)}{|x-y|^s}\, d\mu \\
&\quad + \int_{D_1}\int_{D_2} g\left(\frac{-s_2 w_-(x) - tw_+(y)}{|x-y|^s}\right) \frac{-tw_+(y)}{|x-y|^s}\, d\mu \\
&\quad - \int_{D_2}\int_{D_1} g\left(\frac{tw_+(x) + s_1 w_-(y)}{|x-y|^s}\right) \frac{tw_+(x)}{|x-y|^s}\, d\mu \\
&\quad - \int_{D_1}\int_{D_2} g\left(\frac{-s_1 w_-(x) - tw_+(y)}{|x-y|^s}\right) \frac{-tw_+(y)}{|x-y|^s}\, d\mu \\
&\quad = \int_{D_2}\int_{D_1} \left[g\left(\frac{tw_+(x) + s_2 w_-(y)}{|x-y|^s}\right) - g\left(\frac{tw_+(x) + s_1 w_-(y)}{|x-y|^s}\right)\right] \frac{tw_+(x)}{|x-y|^s}\, d\mu \\
&\quad + \int_{D_1}\int_{D_2} \left[g\left(\frac{-s_2 w_-(x) - tw_+(y)}{|x-y|^s}\right) - g\left(\frac{-s_1 w_-(x) - tw_+(y)}{|x-y|^s}\right)\right] \frac{-tw_+(y)}{|x-y|^s}\, d\mu.
\end{aligned}$$

Hence, since g is increasing, we get

$$\varphi_1(t,s_2) - \varphi_1(t,s_1) \geq 0.$$

Similarly,

$$\varphi_2(t_1, s) - \varphi_2(t_2, s) \geq 0,$$

for $t_1 \leq t_2$ and fixed s. Thus, there are $r, R > 0$, $r < R$ such that

$$\varphi_1(r, s) > 0,\ \varphi_1(R, s) < 0 \quad \text{for all } s \in (r, R],$$

$$\varphi_2(t, r) > 0,\ \varphi_2(t, R) < 0 \quad \text{for all } t \in (r, R],$$

and consequently, there exists $t, s \in [r, R]$ such that $\varphi_1(t, s) = \varphi_2(t, s) = 0$. This shows that $tw_+ - sw_- \in M_3$. □

Lemma 27. *There exist constants $k_j > 0$, $j = 1, 2, 3$, such that for every $u \in K_i$, $i = 1, 2, 3$, there holds*

$$\rho_{s,G}(u) \leq k_1 \int_\Omega f(x, u)u\, dx \leq k_2 \Phi(u) \leq k_3 \rho_{s,G}(u). \tag{28}$$

Proof. Let $u \in K_i$, for some $i = 1, 2, 3$. Then, by definition of the sets K_i,

$$\langle (-\triangle_g)^s u, u\rangle = \int_\Omega f(x, u)u\, dx.$$

Hence,

$$\rho_{s,G}(u) \leq p^- \rho_{s,G}(u) \leq \langle (-\triangle_g)^s u, u\rangle = \int_\Omega f(x, u)u\, dx.$$

Moreover, by $(\mathbf{H}_2)$, it follows that $\mathcal{F}(u)$ is non-negative. Then

$$\Phi(u) = \rho_{s,G}(u) - \mathcal{F}(u) \leq \rho_{s,G}(u).$$

Consequently, by the definitions of the sets K_i and by $(\mathbf{H}_2)$, we have

$$\begin{aligned}
\Phi(u) &= \rho_{s,G}(u) - \mathcal{F}(u) \\
&\geq \frac{1}{p^+} \langle (-\triangle_g)^s u, u\rangle - \int_\Omega F(x, u)\, dx \\
&\geq \frac{1}{p^+} \int_\Omega f(x, u)u\, dx - \frac{1}{c_2} \int_\Omega f(x, u)u\, dx \\
&= \left(\frac{1}{p^+} - \frac{1}{c_2}\right) \int_\Omega f(x, u)u\, dx.
\end{aligned}$$

Therefore, combining the above, we get (28) choosing

$$k_1 = 1,\ k_2 = k_3 = \left[\left(\frac{1}{p^+} - \frac{1}{c_2}\right)\right]^{-1}.$$

□

Lemma 28. *There exists a constant $D > 0$ such that $[u]_{s,G}^{p^+} \geq D$, for all $u \in K_1$, $[u]_{s,G}^{p^+} \geq D$ for all $u \in K_2$, and $[u_-]_{s,G}^{p^+}$, $[u_+]_{s,G}^{p^+} \geq D$ for all $u \in K_3$.*

Proof. Let $u \in K_i$. Suppose, without loss of generality, that $[u_\pm]_{s,G} < 1$. By definition of K_i and hypothesis $(\mathbf{H}_2)$ we have

$$[u_\pm]_{s,G}^{p+} \leq \rho_{s,G}(u_\pm) \leq k_1 \int_\Omega f(x,u)u_\pm\, dx = k_1 \int_\Omega f(x,u_\pm)u_\pm\, dx \leq c\rho_{B,\Omega}(u_\pm).$$

Then using Sobolev inequality in Orlicz fractional spaces

$$\begin{aligned}[u_\pm]_{s,G}^{p+} &\leq c \max\{\|u_\pm\|_{B,\Omega}^{p_B^+}, \|u_\pm\|_{B,\Omega}^{p_B^-}\} \\ &\leq c \max\{[u_\pm]_{s,G}^{p_B^+}, [u_\pm]_{s,G}^{p_B^-}\} = c[u_\pm]_{s,G}^{p_B^-}.\end{aligned}$$

As $p^+ < p_B^-$, the proof is complete. □

Lemma 29. *There exists $c > 0$, $\delta > 0$, $\Phi(u) \geq c[u]_{s,G}^{p^+}$ for every $u \in W_0^{s,G}(\Omega)$ such that $[u]_{s,G} \leq \delta$.*

Proof. Suppose that $[u]_{s,G} \leq 1$, then by Sobolev embedding

$$\begin{aligned}\Phi(u) &= \rho_{s,G}(u) - \mathfrak{F}(u) \geq \rho_{s,G}(u) - c\rho_{B,\Omega}(u) \\ &\geq \rho_{s,G}(u) - c\max\{\|u\|_{B,\Omega}^{p_B^+}, \|u\|_{B,\Omega}^{p_B^-}\} \\ &\geq [u]_{s,G}^{p^+} - c[u]_{s,G}^{p_B^-} \geq c[u]_{s,G}^{p^+}.\end{aligned}$$

If $[u]_{s,G}$ is small, as $p^+ < p_B^-$, the proof is complete. □

Lemma 30. *M_i is a $C^{1,1}$ submanifold of $W_0^{s,G}(\Omega)$ of co-dimension 1 for $i = 1, 2$, and co-dimension 2 for i=3.*

Proof. We define

$$\overline{M}_1 := \left\{ u \in W_0^{s,G}(\Omega) \colon \int_\Omega u_+ dx > 0 \right\},$$

$$\overline{M}_2 := \left\{ u \in W_0^{s,G}(\Omega) \colon \int_\Omega u_- dx > 0 \right\}$$

and

$$\overline{M}_3 := \overline{M}_1 \cap \overline{M}_2.$$

It is enough to prove that M_i is a regular sub-manifold of $W_0^{s,G}(\Omega)$, thanks to the facts that $M_i \subset \overline{M}_i$ and the sets $\overline{M}_i$ are open.

We will define a $C^{1,1}$-function $\varphi_i \colon \overline{M}_i \to \mathbb{R}^d$, where $d = 1$ if $i = 1, 2$ and $d = 2$ if $i = 3$ and M_i will be the inverse image of a regular value of φ_i.

For $u \in \overline{M}_1$,

$$\varphi_1(u) = \langle (-\triangle_g)^s u, u_+ \rangle - \langle \mathcal{F}'(u), u_+ \rangle.$$

Also, for $u \in \overline{M}_2$, let

$$\varphi_2(u) = \langle (-\triangle_g)^s u, u_- \rangle - \langle \mathcal{F}'(u), u_- \rangle.$$

Finally, for $u \in \overline{M}_3$,

$$\varphi_3(u) = (\varphi_1(u), \varphi_2(u)).$$

We want to prove that 0 is a regular value for φ_i. In fact, for $u \in M_1$, we get from Lemma 27 and $(\mathbf{H}_2)$ that

$$\begin{aligned}\langle \varphi_1'(u), u_+\rangle = &\frac{d}{d\varepsilon}\left\langle (-\triangle_g)^s(u+\varepsilon u_+), (u+\varepsilon u_+)_+\right\rangle|_{\varepsilon=0}\\ &-\int_\Omega f(x,u)u_+dx - \int_\Omega f_u(x,u)u_+^2dx.\end{aligned} \tag{29}$$

Observe that

$$\begin{aligned}&\frac{d}{d\varepsilon}\left\langle (-\triangle_g)^s(u+\varepsilon u_+), (u+\varepsilon u_+)_+\right\rangle|_{\varepsilon=0}\\ &\quad= \iint_{\mathbb{R}^{2n}} g'\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\left(\frac{u_+(x)-u_+(y)}{|x-y|^s}\right)^2 d\mu\\ &+\iint_{\mathbb{R}^{2n}} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u_+(x)-u_+(y)}{|x-y|^s}\,d\mu,\end{aligned} \tag{30}$$

where in the last integrals we have used that

$$\frac{d}{d\varepsilon}\left((u+\varepsilon u_+)_+(x)-(u+\varepsilon u_+)_+(y)\right)|_{\varepsilon=0} = \begin{cases}0 & \text{if } u(x),u(y)\le 0,\\ -u(y) & \text{if } u(x)<0, u(y)>0,\\ u(x) & \text{if } u(x)>0, u(y)<0,\\ u(x)-u(y) & \text{if } u(x),u(y)\ge 0,\end{cases}$$

$$= u_+(x)-u_+(y).$$

Hence, by Lemma 10, we get from (30) that

$$\frac{d}{d\varepsilon}\left\langle (-\triangle_g)^s(u+\varepsilon u_+), (u+\varepsilon u_+)_+\right\rangle|_{\varepsilon=0} \le p^+\left\langle (-\triangle_g)^s u, u_+\right\rangle.$$

Thus, we get from (29) that

$$\begin{aligned}\langle \varphi_1'(u), u_+\rangle &\le (p^+-1)\int_\Omega f(x,u)u_+dx - \int_\Omega f_u(x,u)u_+^2dx\\ &\le \left(p^+-1-\frac{1}{c_3}\right)\int_\Omega f(x,u)u_+dx,\end{aligned} \tag{31}$$

where $p^+-1-\dfrac{1}{c_3}<0$ by $(\mathbf{H}_2)$. Now, by Lemma 27, we obtain

$$\langle \varphi_1'(u), u_+\rangle \le -C\rho_{s,G}(u_+)$$

which is strictly negative by Lemma 28. Then $\varphi_1'(u)\ne 0$ and we have $M_1 = \varphi_1^{-1}(0)$ is a smooth sub-manifold of $W_0^{s,G}(\Omega)$.

In a similar way we can prove that M_2 is also a smooth sub-manifold of $W_0^{s,G}(\Omega)$.

We next consider M_3. For $u \in M_3$, we have

$$\langle \varphi_1'(u), u\rangle = \iint_{\mathbb{R}^{2n}} g'\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{u_+(x)-u_+(y)}{|x-y|^s}\,d\mu$$
$$+\iint_{\mathbb{R}^{2n}} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u_+(x)-u_+(y)}{|x-y|^s}d\mu - \int_\Omega f(x,u)u_+dx - \int_\Omega f_u(x,u)u_+^2\,dx.$$

Appealing to the fact that

$$sign(a-b) = sign(a_+ - b_+),$$

for all $a, b \in \mathbb{R}$ such that $a-b, a_+ - b_+ \neq 0$, and using (6), it follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} g'\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{u_+(x)-u_+(y)}{|x-y|^s}\,d\mu\\
&\quad+\iint_{\mathbb{R}^{2n}} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u_+(x)-u_+(y)}{|x-y|^s}\,d\mu\\
&\quad\le p^+\iint_{\mathbb{R}^{2n}} g\left(\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{u_+(x)-u_+(y)}{|x-y|^s}d\mu\\
&\quad= p^+\left\langle(-\triangle_g)^s u, u_+\right\rangle.
\end{aligned}$$

Hence, recalling that $u \in M_1$ and proceeding as in (31), we obtain that

$$\langle \varphi_1'(u), u\rangle < 0.$$

Similarly, by the fact that

$$sign(a-b) \neq sign(a_- - b_-),$$

for all $a, b \in \mathbb{R}$ such that $a-b, a_- - b_- \neq 0$, using (6) and the choice of c_3 in ($\mathbf{H}_2$), it follows that

$$\langle \varphi_2'(u), u\rangle \le p^-\left\langle(-\triangle_g)^s u, u_+\right\rangle - \int_\Omega f(x,u)u_-dx - \int_\Omega f_u(x,u)u_-^2dx < 0.$$

Therefore M_3 is a smooth sub-manifold of $W_0^{s,G}(\Omega)$. □

Lemma 31. *The sets K_i are complete.*

Proof. We consider K_3, the proofs for K_1 and K_2 are similar. Let $\{u_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in K_3, then $u_k \to u$ in $W_0^{s,G}(\Omega)$. In particular, $(-\triangle_g)^s u_k \to (-\triangle_g)^s u$ in $W^{-s,\widetilde{G}}(\Omega)$ and $u_{k_+} \rightharpoonup u_+$ in $W_0^{s,G}(\Omega)$. So,

$$\begin{aligned}
&\lim_{k\to\infty}\left\langle(-\triangle_g)^s u_k, u_{k_+}\right\rangle\\
&\quad= \lim_{k\to\infty}\left[\left\langle(-\triangle_g)^s u_k - (-\triangle_g)^s u, u_{k_+}\right\rangle + \left\langle(-\triangle_g)^s u, u_{k_+}\right\rangle\right]\\
&\quad= \left\langle(-\triangle_g)^s u, u_+\right\rangle.
\end{aligned}\tag{32}$$

Moreover, using the compact embedding $W_0^{s,G}(\Omega) \hookrightarrow L^B(\Omega)$, we get $u_{k_+} \to u_+$ in $L^B(\Omega)$ and hence by ($\mathbf{H}_0$), we obtain

$$\int_\Omega f(x,u_k)u_{k_+}dx \to \int_\Omega f(x,u)u_+dx.$$

Hence, recalling Lemma 28, we get $u_+ \in M_1$. Similarly, $u_- \in M_2$, and thus we obtain $u \in K_3$. This shows that K_3 is complete. □

For the next result, we recall the definition of tangent space of Banach manifolds.

Definition 32. *Let X be a Banach space and $M \subset X$ be a Banach manifold. The tangent space at u of M is*

$$T_uM = \{v \colon \exists \alpha : (-1,1) \to M \quad \alpha(0) = u \quad \textit{and} \quad \alpha'(0) = v\}.$$

We recall that in Banach manifolds, T_uM is a closed linear subspace of X.

Lemma 33. *For every $u \in M_i$ we obtain the direct decomposition*

$$T_uW_0^{s,G}(\Omega) = T_uM_i \oplus \mathit{span}\{u_+, u_-\}.$$

Moreover, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. We show the descomposition of M_1. Let $v \in T_uW_0^{s,G}(\Omega)$ and $v = v_1 + v_2$, where $v_2 = \alpha u_+$ and $v_1 = v - v_2$. We are interesting in choosing α so that $v_1 \in T_uM_1$,

$$\langle \varphi_1'(u), v\rangle = \langle \varphi_1'(u), \alpha u_+\rangle + \langle \varphi_1'(u), v_1\rangle.$$

If we choose

$$\alpha = \frac{\langle \varphi_1'(u), v\rangle}{\langle \varphi_1'(u), u_+\rangle},$$

then $\langle \varphi_1'(u), v_1\rangle = 0$. So, we obtain

$$T_uW_0^{s,G}(\Omega) = T_uM_1 \oplus \mathit{span}\{u_+\},$$

where $M_1 = \{u \colon \varphi_1(u) = 0\}$ and $T_uM_1 = \{v \colon \langle \varphi_1'(u), v\rangle = 0\}$. Analogously,

$$T_uW_0^{s,G}(\Omega) = T_uM_2 \oplus \mathit{span}\{u_-\}$$

and

$$T_uW_0^{s,G}(\Omega) = T_uM_3 \oplus \mathit{span}\{u_+, u_-\}.$$

This finishes the proof. □

In the next lemma, we prove that the unrestricted functional Φ satisfies the Palais-Smale condition, that is, if $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{s,G}(\Omega)$ is a sequence such that $\Phi(u_j)$ is uniformly bounded and $\Phi'(u_j) \to 0$ strongly in $W^{-s,\widetilde{G}}(\Omega)$, then $\{u_j\}_{j\in\mathbb{N}}$ contains a strongly convergent subsequence. The arguments of the proof are standard. However, we write it for convenience of the reader.

Lemma 34. *The unrestricted functional Φ satisfies the Palais-Smale condition.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{s,G}(\Omega)$ be a Palais-Smale sequence, i.e $\Phi(u_j)$ is uniformly bounded and $\Phi'(u_j) \to 0$ strongly in $W^{-s,\widetilde{G}}(\Omega)$. Since $\Phi(u_j)$ is uniformly bounded, using Lemma 27, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{s,G}(\Omega)$.

We define $\Phi'(u_j) := \psi_j$, then

$$\langle \Phi'(u_j), z\rangle = \langle \psi_j, z\rangle, \ \textit{for all } z \in W_0^{s,G}(\Omega).$$

Also,

$$\langle \Phi'(u_j), z\rangle = \iint_{\mathbb{R}^{2n}} g\left(\frac{u_j(x)-u_j(y)}{|x-y|^s}\right)\frac{z(x)-z(y)}{|x-y|^s}d\mu - \int_\Omega f(x,u_j)z dx.$$

Then, v is a weak solution of problem

$$\begin{cases} (-\triangle_g)^s v = f(x,u_j)+\psi_j & \text{in } \Omega,\\ v=0 & \text{in } \Omega^c.\end{cases}$$

Observe that by $(\mathbf{H}_0)$, $f_j := f(x,u_j)+\psi_j \in W^{-s,\widetilde{G}}(\Omega)$.

Now, we define the operator $T\colon W^{-s,\widetilde{G}}(\Omega)\to W_0^{s,G}(\Omega)$, as $T(h):=u$, where u is a weak solution of problem

$$\begin{cases} (-\triangle_g)^s u = h & \text{in } \Omega,\\ u=0 & \text{in } \Omega^c.\end{cases}$$

By Lemma 11 we know that $(-\triangle_g)^s$ is continuous and that admits a continuous inverse on $W^{-s,\widetilde{G}}(\Omega)$. It is enough to see that $f_j \to f(\cdot,u(\cdot))$ in $W^{-s,\widetilde{G}}(\Omega)$. As $\Phi'(u_j)\to 0$ strongly in $W^{-s,\widetilde{G}}(\Omega)$, we only need to check that $\{f(x,u_j)\}_{j\in\mathbb{N}}$ has a subsequence that strongly converges in $W^{-s,\widetilde{G}}(\Omega)$, the proof of this fact can be found in [3, Lemma 3.5]. This finishes the proof.

□

In the sequel, we will need the following fact from [21, Lemma 4.4]:

$$\Phi(u) > \Phi(u_+) + \Phi(-u_-), \tag{33}$$

for all $u\in W^{s,G}(\mathbb{R}^n)$ with $u_\pm \neq 0$.

The following result follows exactly as in [13, Lemma 5], using Lemma 33 together with (33), Lemma 28 with Poincaré inequality (see Proposition 3.2 in [24]), the previous lemma and the fact that M_i are complete manifolds.

Lemma 35. *The restricted functionals $\Phi\mid_{M_i}$ satisfy the Palais-Smale condition.*

Moreover, we get as in [26]

Lemma 36. *Let $u\in M_i$ be a critical point of the restricted functional $\Phi\mid_{M_i}$. Then u is also a critical point of the unrestricted functional Φ and hence a weak solution to the problem* (1) *where f verifies* $(\mathbf{H}_0)-(\mathbf{H}_2)$.

Finally, we give the proof of the existence of two constant sign solutions and a nodal solution to problem (1). Here, we mainly follow [19].

Theorem 37. *Under assumptions* $(\mathbf{H}_0)$–$(\mathbf{H}_2)$*, there exists three different, nontrivial, weak solutions of problem* (1)*. Moreover, we have that one is positive, one is negative and the other one has non-constant sign.*

Proof. From Lemma 29 and Poincaré inequality (Proposition 3.2 in [24]), there exists a constant $0<c<1$ such that

$$\Phi(u)\geq c[u]_{s,G}^{p^+}$$

when $[u_-]_{s,G} < c$. By continuity of the projection $P(u) = u_-$, the set $U = \{u \in M_1 : [u_-]_{s,G} < c\}$ is open, contains K_1 and its closure $\overline{U}$ is complete since M_1 is complete and $\overline{U}$ is closed. Since Φ is bounded from below in $\overline{U}$, we let

$$m := \inf_{\overline{U}} \Phi(u).$$

Take u_j a minimizing sequence. By (33), we get that $u_{j,+}$ is also a minimizing sequence for Φ and moreover $u_{j,+} \in K_1$. For all j large enough, take $\varepsilon_j > 0$ such that

$$\Phi(u_{j,+}) < m + \varepsilon_j, \quad \varepsilon < c^{p_+ +1}.$$

Also, put $\delta_j := \sqrt{\varepsilon_j}$. By the Ekeland Variational Principle, there is a sequence $v_j \in \overline{U}$ such that

$$[u_{j,+} - v_j]_{s,G} \leq \delta_j, \quad \Phi(v_j) \leq \Phi(u_{j,+}) < m + \varepsilon_j \tag{34}$$

and

$$\Phi(v_j) < \Phi(w) + \frac{\varepsilon_j}{\delta_j}[v_j - w]_{s,G}, \quad w \in \overline{U}, w \neq v_j. \tag{35}$$

We will prove next that $v_j \in U$. To get a contradiction, assume that $[v_{j,-}]_{s,G} = c$. Then,

$$\Phi(-v_{j,-}) \geq c[v_{j,-}]_{s,G}^{p^+} = c^{p^+ +1} > \varepsilon_j.$$

On the other hand, since $v_{j,+} \in K_1$, we have from (33)

$$\Phi(v_j) > \Phi(v_{j,+}) + \Phi(-v_{j,-}) \geq m + \varepsilon_j,$$

which contradicts (34). Thus, $v_j \in U$. Next, since $v_j \in U$ with U open, by (35), there holds $(\Phi_{|M_1})'(v_j) \to 0$. By Lemma 35, v_j contains a convergence subsequence, also denoted v_j, with limit v. From (34), we get $u_{j,+} \to v$, and from the completeness of K_1, it follows $v \in K_1$ and so $v \in U$ (v is in the interior of M_1). Moreover, by continuity, $(\Phi_{|M_1})'(v) = 0$. By Lemma 36, v is a critical point of Φ as well. Similarly, we can state the result for K_2 and K_3. □

CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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