AN EXISTENCE RESULT FOR A SCHRÖDINGER-KIRCHHOFF CRITICAL PROBLEM IN p-MAGNETIC FRACTIONAL SOBOLEV SPACES

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Abstract. In this work, we study the existence of global and non-trivial weak solutions to the following problem with critical growth and involving the p-fractional magnetic Laplacian $(-\Delta_p^A)^s$:

$$M\left([u]_{s,p}^A\right)(-\Delta_p^A)^s u + V(x)|u|^{p-2}u = |u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \quad N \geq 3.$$

Here M is a Kirchhoff function, V is a scalar potential, and $p_s^* = Np/(N-sp)$ is the critical fractional Sobolev exponent. The solvability is proved by appealing to critical point theory, without the Palais-Smale condition, under a careful analysis of the fractional magnetic gradients and the critical term. To treat this latter contribution, we develop a concentration compactness principle for bounded sequences in appropriate magnetic Sobolev spaces.

1. Introduction

In this paper, we establish a concentration compactness principle for magnetic fractional Sobolev spaces and we apply it to study global weak solutions to the following critical-growth problem

$$M\left([u]_{s,p}^A\right)(-\Delta_p^A)^s u + V(x)|u|^{p-2}u = |u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $1 < p < p_s^*$, and $p_s^* = Np/N - sp$ is the critical Sobolev exponent in the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. Moreover, the leading operator $(-\Delta_p^A)^s$ is the p-fractional magnetic operator defined for a sufficiently smooth function u and up to a multiplicative constant as

$$(-\Delta_p^A)^s u(x) := 2\text{P.V.} \int_{\mathbb{R}^N} |D_{s,A}u(x,y)|^{p-2} D_{s,A}u(x,y) \frac{dy}{|x-y|^{N+s}} \quad x \in \mathbb{R}^N, \tag{1.2}$$

with $A : \mathbb{R}^N \to \mathbb{R}^N$ is a given vector field, and

$$D_{s,A}u(x,y) = \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^s}.$$

In addition, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential, and $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a Kirchhoff function. Physical motivation for studying the operator (1.2) are given in [25] and [26] (see also the introduction of [16]).

Equation (1.1) has different contributions. Firstly, the operator $(-\Delta_p^A)^s$ arises as a non-local generalization of the (real part of the) magnetic Schrödinger operator defined as

$$-(\nabla - iA)^2 u(x) = -\Delta u(x) + 2iA(x) \cdot \nabla u + |A(x)|^2 u + iu \, \text{div} A(x). \tag{1.3}$$

In this context, the field $B = \nabla \times A$ represent a magnetic field acting on a charged particle, like an electron. When $N = 3$, B is the usual curl operator. For general N, $B = (B_{jk})$ where

$$B_{jk} := \partial_j A_k - \partial_k A_j.$$

When there is also a conservative electric field acting on the particle, the scalar function V represents its potential. The magnetic Schrödinger operator has been studied extensively in the last decades. We refer the reader to the references [6], [8], [15], and [36], among many others. Regarding the Kirchhoff term $M\left([u]_{s,p}^A\right)$, we first recall that Kirchhoff [27] introduced a model for a vibrating string given by

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right| \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where L is the length of the string, h the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and ρ_0 the initial tension. That equation constitutes an extension of the D'Alembert's classical wave equation, since it takes into account the changes in length of the string under vibration. There have been further generalizations of the Kirchhoff model. In particular, we quote the nice article [23], where the authors studied the existence of non-negative weak solutions for a nonlocal Kirchhoff type problem of the form

$$M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2^*-2}u.$$

In this case, the tension M depends on the fractional length of the string.

A pioneering work treating local elliptic problem with critical exponent is [13]. Afterwards, many papers have considered similar problems in different context. However, in this brief review, we will focus on those for nonlocal problems in the magnetic framework. The first work considering the fractional magnetic operator with lower order terms is [16] (in the spirit of the seminal work [18]). In this work, the authors proved existence of global weak solutions for minimization problems in the subcritical case, under constraints and involving the operator (1.2) for $p = 2$ and $N = 3$. The existence is provided for radial Sobolev functions or under further assumptions on the magnetic potential A. In the critical case, they did not show existence, but they gave a representation of solutions similar to the case $A = 0$ and proved some nonexistence results. Multiplicity results for subcritical problems in bounded domains are given in [22], while for unbounded domain are provided in [34] for the p-fractional magnetic operator, and in [31] in the more general context of Orlicz-Sobolev spaces.

Problems depending on a parameter $\varepsilon > 0$ of the form

$$M_\varepsilon([u]^2_{A,s})(-\Delta)^s_{\varepsilon^{-1}A}u + V(x)u = f(x,|u|)u + K(x)|u|^{r-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

have been also considered extensively in the literature by appealing to different techniques like penalization, Nehari method, and Ljusternik-Schnirelman theory, among many others (see for instance [1], [2], [3], [4], [10], [37], and the references therein). We point out that in the previous list of references, the subcritical, critical and supercritical cases were studied for (1.4), varying also the assumptions on M, the potential V, and the nonlinearity f. The interest in solutions u_ε of (1.4) and their behaviour as $\varepsilon \to 0$ are motivated from the fact that the transition from quantum mechanics to classical mechanics can be performed formally in the time-dependent Schrödinger equation letting $\varepsilon \to 0$. We refer the reader to the nice book [5] where a complete review of recent references on the subject is given.

The energy functional $I : \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N) \to \mathbb{R}$ associated to Equation (1.1) is given by

$$I(u) = \frac{1}{p}\mathcal{M}\left([u]^A_{s,p}\right) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \frac{1}{p^*_s}\int_{\mathbb{R}^N} |u|^{p^*_s}\,dx.$$

To get existence of solutions to (1.1) we will look for critical points of the functional I. However, the functional does not satisfy in general the Palais-Smale condition due to the presence of the critical exponent in the right-hand side of (1.1). To overcome this difficulty we develop a concentration compactness principle based on Lions's results [28] adapted to magnetic fractional Sobolev spaces. The Lions' result has been extended to different contexts see for example [19], [30] and [33]. The statement is the following:

Theorem 1.1. *Let $\{u_k\} \subset \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ be a weakly convergent sequence with limit $u \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$. Then there exist: two bounded measures μ and ν, an at most countable set J, and positive real numbers η_j, ν_j, $j \in J$, such that*

$$|D^A_s u_k|^p\,dx \rightharpoonup \mu \geq |D^A_s u|^p\,dx + \sum_j \mu_j \delta_{x_j}, \tag{1.5}$$

$$|u_k|^{p^*_s}\,dx \rightharpoonup \nu = |u|^{p^*_s}\,dx + \sum_j \nu_j \delta_{x_j}, \tag{1.6}$$

and

$$S_A^{1/p}\nu_j^{1/p_s^*} \leq \mu_j^{1/p}, \quad \textit{for all } j \in J, \tag{1.7}$$

where

$$S_A := \inf_{u\in\mathcal{D}_A^{s,p}(\mathbb{R}^N),u\neq 0} \frac{[u]_{s,p}^A}{\|u\|_{p_s^*}^p} > 0.$$

We refer the reader to Section 2 for the function notation and a discussion of S_A.

Afterwards, we will apply a version of the mountain pass theorem [7] to obtain a Palais-Smale (PS-) sequence. The lack of compactness prevents us to prove the strong convergence of the sequence. However, the presence of the potential V and the diamagnetic inequality (2.4) will allow us to apply compactness results from [29] into real Sobolev fractional spaces. These results, together with the concentration compactness principle and a careful treatment of the magnetic fractional gradients, will help us to state local convergence of the PS-sequence in the critical Sobolev spaces $L^{p_s^*}$ and pointwise convergence of the magnetic fractional gradients. In this way we obtain the second main result of the paper (we refer the reader to Section 2 for a discussion of the assumptions).

Theorem 1.2. *Assume* $(HM)_1$–$(HM)_3$ *on* M *and* (V) *on the potential* V*. Suppose also that* $A \in L^\infty_{loc}(\mathbb{R}^N, \mathbb{R}^N)$*. Then, the Equation* (1.1) *has a non-trivial global weak solution* $u \in \mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N)$*.*

The paper is organized as follows. In Section 2 we will provide the basic assumptions, definitions and results used in the manuscript. In Section 3, we will gain some insight into the fractional magnetic operator by giving sufficient conditions for $(-\Delta_p^A)^s$ to be finite, and the relation between weak and pointwise solutions of (1.1). Afterwards, in Section 4, we will provide the concentration compactness principle for our framework, while in Sections 5 and 6, we will give the full proof of Theorem 1.2. This latter achievement will be done in a series of steps.

2. Preliminaries

2.1. **Main assumptions.** For the Kirchhoff function $M = M(t) : \mathbb{R}_+ \to \mathbb{R}_+$, we will suppose the following:

$(HM)_1$ M is continuous and for any $\alpha > 0$, there exists $\delta = \delta(\alpha)$ such that

$$M(t) \geq \delta, \text{ for all } t \geq \alpha, \quad t \geq 0.$$

$(HM)_2$ Let

$$\mathcal{M}(t) := \int_0^t M(s)\, ds.$$

There exists $\theta \in (1, p_s^*/p)$ such that

$$M(t)t \leq \theta \mathcal{M}(t), \qquad \text{for all } t \geq 0.$$

$(HM)_3$ There is $c_0 \in (0,1)$ such that for all $t \in [0,1]$ there holds

$$M(t) \geq c_0 t^{\theta - 1}.$$

The assumptions on the potential $V = V(x)$ are the following:

(V) The function $V : \mathbb{R}^N \to \mathbb{R}$ is non-negative, continuous in $\mathbb{R}^N$ and satisfies

- $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;
- There exists $h > 0$ such that for all $c > 0$,

$$\lim_{|y| \to \infty} |\{x \in B_h(y) : V(x) \leq c\}| = 0.$$

Remark 2.1. The above assumptions for V are taken from the reference [35] in order to guarantee continuity and compactness of inclusions between weighted Sobolev spaces and Lebesgue spaces. See next section for further details.

2.2. **Functional framework.** Define the Lebesgue space for $p > 1$,

$$L^p(\mathbb{R}^N, \mathbb{C}) := \left\{ u : \mathbb{R}^N \to \mathbb{C}, \int_{\mathbb{R}^N} |u(x)|^p \, dx < \infty \right\},$$

and, for a non-negative $V : \mathbb{R}^N \to \mathbb{R}$, the weighted Lebesgue space

$$L^p_V(\mathbb{R}^N, \mathbb{C}) := \left\{ u : \mathbb{R}^N \to \mathbb{C}, \int_{\mathbb{R}^N} |u(x)|^p V(x) \, dx < \infty \right\}.$$

For $s \in (0,1)$, denote by $\mathcal{D}^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{s,p} := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{sp}} \, d\eta, \qquad d\eta := \frac{dx\, dy}{|x - y|^N},$$

that is

$$\mathcal{D}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty \right\}.$$

We will also use the next notation for fractional gradients of real functions φ:

$$D_s\varphi(x, y) := \frac{\varphi(x) - \varphi(y)}{|x - y|^s} \tag{2.1}$$

and

$$|D^s\varphi(x)|^p := \int_{\mathbb{R}^N} |D_s\varphi(x, y)|^p \, \frac{dy}{|x - y|^N}. \tag{2.2}$$

For a given magnetic potential $A \in L^\infty_{loc}(\mathbb{R}^N, \mathbb{R}^N)$, we consider fractional magnetic spaces:

$$\mathcal{D}^{s,p}_A(\mathbb{R}^N, \mathbb{C}) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N, \mathbb{C}) : [u]^A_{s,p} < \infty \right\}$$

where

$$[u]^A_{s,p} := \int_{\mathbb{R}^{2N}} |D_{s,A}u(x, y)|^p d\eta,$$

with

$$D_{s,A}u(x,y) := \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^s}.$$

We will also use the following notation for magnetic fractional gradients

$$|D_s^A u(x)|^p := \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^p}{|x-y|^{sp+N}}\, dy, \tag{2.3}$$

so that

$$[u]_{s,p}^A = \int_{\mathbb{R}^N} |D_s^A u(x)|^p\, dx.$$

The space $\mathcal{D}_A^{s,p}(\mathbb{R}^N, \mathbb{C})$ is equipped with the norm

$$\|u\|_{s,p}^A = ([u]_{s,p}^A + \|u\|_{p_s^*}^p)^{1/p}.$$

By (2.5) below, we obtain that

$$\|u\| := ([u]_{s,p}^A)^{1/p}$$

is an equivalent norm on $\mathcal{D}_A^{s,p}(\mathbb{R}^N, \mathbb{C})$.

Finally, we consider the weighted fractional magnetic space of order s,

$$\mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N, \mathbb{C}) := \left\{ u \in \mathcal{D}_A^{s,p}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u(x)|^p V(x)\, dx < \infty \right\}$$

with the norm

$$\|u\|_{s,p,V}^A := \left((\|u\|_{s,p}^A)^p + \int_{\mathbb{R}^N} |u(x)|^p V(x)\, dx \right)^{1/p}.$$

Observe that since $V(x) \geq V_0 > 0$, we get $u \in L^p(\mathbb{R}^N)$. Moreover, we obtain that $\|u\|_{s,p,V}^A$ is equivalent to

$$\|u\| = \left(\int_{\mathbb{R}^N} |u(x)|^p V(x)\, dx + [u]_{s,p}^A \right)^{1/p}.$$

We will use this norm in the sequel.

For further reference, we introduce the best Sobolev constant in the embedding $\mathcal{D}^{s,p}_A(\mathbb{R}^N,\mathbb{C}) \hookrightarrow L^{p^*_s}(\mathbb{R}^N,\mathbb{C})$. Let

$$0 < S := \inf_{u \in \mathcal{D}^{s,p}(\mathbb{R}^N,\mathbb{R}), u \neq 0} \frac{[u]_{s,p}}{\|u\|^p_{p^*_s}}.$$

By the diamagnetic inequality (for the proof see for instance [16])

$$||u(x)| - |u(y)|| \leq \left| u(x) - e^{i(x-y)A(\frac{x+y}{2})} u(y) \right|, \tag{2.4}$$

if $u \in \mathcal{D}^{s,p}_A(\mathbb{R}^N,\mathbb{C})$, then $|u| \in \mathcal{D}^{s,p}(\mathbb{R}^N,\mathbb{R})$, hence by [17, Theorem 6.5]

$$S\|u\|^p_{p^*_s} = S\||u|\|^p_{p^*_s} \leq [|u|]_{s,p} \leq [u]^A_{s,p}.$$

Thus,

$$S_A := \inf_{u \in \mathcal{D}^{s,p}_A(\mathbb{R}^N,\mathbb{C}), u \neq 0} \frac{[u]^A_{s,p}}{\|u\|^p_{p^*_s}} \geq S > 0. \tag{2.5}$$

We will usually omit $\mathbb{R}$ or $\mathbb{C}$ in the notation of Lebesgue and Sobolev spaces. The real or complex case will be clear from the context.

2.3. **Weak solutions.** We next introduce the notion of weak solutions to (1.1).

Definition 2.2. *We say that $u \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ is a weak solution of (1.1) if*

$$Re\left[M([u]^A_{s,p}) \int_{\mathbb{R}^{2N}} |D_{s,A}u|^{p-2} D_{s,A}u\overline{D_{s,A}\varphi}\, d\eta + \int_{\mathbb{R}^N} |u|^{p-2}u\overline{\varphi}V(x)\, dx\right] = Re\left[\int_{\mathbb{R}^N} |u|^{p^*_s-2}u\overline{\varphi}\, dx\right],$$

for any $\varphi \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$.

We point out that by Hölder inequality and the facts that $u \in L^p_V(\mathbb{R}^N) \cap L^{p^*_s}(\mathbb{R}^N)$, all the terms in the above definition are finite.

Finally, we say that u is a pointwise solution of (1.1) if it satisfies the equation in the pointwise sense for a. e. x in $\mathbb{R}^N$. In particular, $(-\Delta^A_p)^s u < \infty$ a.e. in $\mathbb{R}^N$.

3. Preliminaries on the p-fractional magnetic Laplacian

In this section, we provide regularity conditions on u in order for $(-\Delta_p^A)^s u$ to be finite a.e. in $\mathbb{R}^N$. The discussion is inspired by [16]. However, unlike [16], we will not impose the boundedness of u. Instead, we will consider a special class of locally integrable functions, called tail space, defined as

$$L_s^{p-1}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{1+|x|^{N+sp}}\, dx < \infty \right\}.$$

Throughout this section, we will use the following notation for a function $u : \mathbb{R}^N \to \mathbb{C}$ and $x \in \mathbb{R}^N$:

$$u_x(y) := e^{i(x-y)A(\frac{x+y}{2})} u(y), \quad y \in \mathbb{R}^N.$$

Theorem 3.1. *Assume $A \in C^2(\mathbb{R}^N)$ and that $u \in L_s^{p-1}(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$, with $\alpha \in (0,1)$ such that $\alpha > \max\{1-p(1-s), 0\}$ if $p \geq 2$, and $\alpha > \max\{1-p(1-s), 0\}/(p-1)$ if $1 < p < 2$. Then,*

$$(-\Delta_p^A)^s u(x) < \infty, \quad \textit{a.e. in } \mathbb{R}^N.$$

Proof. For $0 < \varepsilon < 1$, define

$$g(x) := \int_{\mathbb{R}^N \setminus B_1(x)} \frac{|u_x(x) - u_x(y)|^{p-2}(u_x(x) - u_x(y))}{|x-y|^{N+sp}} dy$$

and

$$g_\varepsilon(x) := \int_{B_1(x)} \frac{|u_x(x) - u_x(y)|^{p-2}(u_x(x) - u_x(y))}{|x-y|^{N+sp}} \chi_{\varepsilon,x}(y) dy,$$

where $\chi_{\varepsilon,x}$ is the characteristic function of $B_1(x) \setminus B_\varepsilon(x)$.

We first treat the term g. Observe that by [20, Lemma A. 5], there exists $C > 0$ such that

$$|x-y| \geq C(1+|y|), \quad y \in \mathbb{R}^N \setminus B_1(x).$$

Moreover, since $u \in L_s^{p-1}(\mathbb{R}^N)$, we have

(3.1)

$$|g(x)| \leq \int_{\mathbb{R}^N\setminus B_1(x)} \frac{|u_x(x)-u_x(y)|^{p-1}}{|x-y|^{N+sp}}\,dy \leq C\left(\int_{\mathbb{R}^N\setminus B_1(x)} \frac{|u(x)|^{p-1}}{1+|y|^{N+sp}}\,dy + \int_{\mathbb{R}^N\setminus B_1(x)} \frac{|u(y)|^{p-1}}{1+|y|^{N+sp}}\,dy\right) < \infty,$$

where we have also used that the constant function $|u(x)|$ is also in $L^{p-1}_s(\mathbb{R}^N)$.

Regarding the second term g_ε, we write

$$g_\varepsilon(x) = \frac{1}{2}\int_{B_1(x)} \frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))}{|x-y|^{N+sp}}\chi_{\varepsilon,x}(y)dy + \frac{1}{2}\int_{B_1(x)} \frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))}{|x-y|^{N+sp}}\chi_{\varepsilon,x}(y)dy,$$

and we perform the change of variables $h = y-x$ in the first integral, and $h = x-y$ in the second one, to get

$$\begin{aligned}
g_\varepsilon(x) &= \frac{1}{2}\int_{B_1(0)} \frac{|u_x(x)-u_x(x+h)|^{p-2}(u_x(x)-u_x(x+h))}{|h|^{N+sp}}\chi_\varepsilon(h)dh \\
&+ \frac{1}{2}\int_{B_1(0)} \frac{|u_x(x)-u_x(x-h)|^{p-2}(u_x(x)-u_x(x-h))}{|h|^{N+sp}}\chi_\varepsilon(h)dh \\
&= \frac{1}{2}\int_{B_1(0)} \left(\frac{|u_x(x)-u_x(x+h)|^{p-2}(u_x(x)-u_x(x+h)) + |u_x(x)-u_x(x-h)|^{p-2}(u_x(x)-u_x(x-h))}{|h|^{N+sp}}\right)\chi_\varepsilon(h)dh.
\end{aligned}$$

where now χ_ε is the characteristic function of $B_1(0)\setminus B_\varepsilon(0)$. Let $J(\xi) = |\xi|^{p-2}\xi$, $\xi\in\mathbb{R}$. Using the inequalities ([9, Lemma 2.4])

$$|J(a)-J(b)| \leq \begin{cases} C_p|a-b|(|a|^{p-2}+|b|^{p-2}) & \text{if } p\geq 2 \\ C'_p|a-b|^{p-1} & \text{if } 1<p\leq 2 \end{cases}, \quad a,b\in\mathbb{R},$$

we obtain for $0<|h|<1$,

(3.2)

$$\begin{aligned}
&\Big||u_x(x)-u_x(x+h)|^{p-2}(u_x(x)-u_x(x+h)) - |u_x(x)-u_x(x-h)|^{p-2}(u_x(x-h)-u_x(x))\Big| \\
&\leq \begin{cases} C_p|2u_x(x)-u_x(x-h)-u_x(x+h)|(|u_x(x)-u_x(x-h)|^{p-2}+|u_x(x)-u_x(x+h)|^{p-2}) & \text{if } p\geq 2 \\ C'_p|2u_x(x)-u_x(x-h)-u_x(x+h)|^{p-1} & \text{if } 1<p\leq 2. \end{cases}
\end{aligned}$$

By [16, Lemma 2.5], we have that

$$|2u_x(x) - u_x(x-h) - u_x(x+h)| \leq C|h|^{1+\alpha}. \tag{3.3}$$

Therefore, by (3.2) and the assumption that u is locally bounded and C^1_{loc}, we have for $0 < |h| < 1$

$$\left|\frac{|u_x(x) - u_x(x+h)|^{p-2}(u_x(x) - u_x(x+h)) + |u_x(x) - u_x(x-h)|^{p-2}(u_x(x) - u_x(x-h))}{|h|^{N+sp}}\right|$$

$$\leq \begin{cases} C_p|h|^{1+\alpha+p-2-N-sp} & \text{if } p \geq 2 \\ C'_p|h|^{(1+\alpha)(p-1)-N-sp} & \text{if } 1 < p \leq 2. \end{cases} \tag{3.4}$$

Observe that the right-hand side of (3.4) is integrable in $B_1(0)$ by the assumptions on α. Hence, by dominated convergence theorem,

$$(-\Delta_p^A)^s u(x) = g(x) + \lim_{\varepsilon \to 0} g_\varepsilon(x) < \infty \quad a.e.$$

■

Remark 3.2. Observe that as $p \to 2$ from above or below, the lower bound in the Hölder exponent α goes to $2s - 1$, which is the same bound in [16] for $p = 2$.

We observe that the constant C in (3.3) is the same for all x in a compact set K. Moreover, since u is locally bounded and $u \in L_s^{p-1}(\mathbb{R}^N)$, we get $|g(x)| \leq C$ for all $x \in K$. Also, $|g_\varepsilon(x)| \leq C$ for all $\varepsilon > 0$ and $x \in K$ by (3.4). Hence,

$$g + g_\varepsilon \to (-\Delta_p^A)^s u \text{ in } L^1(K), \ \text{ as } \varepsilon \to 0. \tag{3.5}$$

We use this fact to prove the following relation between weak and pointwise solutions of (1.1).

Theorem 3.3. *Assume that $A \in C^2(\mathbb{R}^N)$ and that $u \in L_s^{p-1}(\mathbb{R}^N) \cap C_{loc}^{1,\alpha}(\mathbb{R}^N)$, with $\alpha \in (0,1)$ such that $\alpha > \max\{1-p(1-s),0\}$ if $p \geq 2$, and $\alpha > \max\{1-p(1-s),0\}/(p-1)$ if $1 < p < 2$. If $u \in \mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N,\mathbb{C})$ is a weak solution of (1.1), then it is a pointwise solution in $\mathbb{R}^N$.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} |D_{s,A}u|^{p-2} D_{s,A}u \overline{D_{s,A}\varphi}\, d\eta \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2N}} \frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))\overline{\varphi_x(x)-\varphi_x(y)}}{|x-y|^{N+sp}} \chi_\varepsilon(y)\, dy\, dx, \quad (\chi_\varepsilon := \chi_{\mathbb{R}^N\setminus B_\varepsilon(x)}) \\
&= \lim_{\varepsilon\to 0} \left[\int_{\mathbb{R}^N} (g(x)+g_\varepsilon(x))\overline{\varphi(x)}\, dx - \int_{\mathbb{R}^{2N}} \frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))\overline{\varphi_x(y)}}{|x-y|^{N+sp}} \chi_\varepsilon(y)\, dy\, dx \right].
\end{aligned} \tag{3.6}$$

We observe that since φ is bounded, $u \in L_s^{p-1}(\mathbb{R}^N)$, and for $\varepsilon > 0$ fixed, the term

$$\frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))\overline{\varphi_x(y)}}{|x-y|^{N+sp}} \chi_\varepsilon(y)$$

is integrable in $\mathbb{R}^{2N}$. Hence, by Fubini Theorem, we get

$$\int_{\mathbb{R}^{2N}} \frac{|u_x(x)-u_x(y)|^{p-2}(u_x(x)-u_x(y))\overline{\varphi_x(y)}}{|x-y|^{N+sp}} \chi_\varepsilon(y)\, dy\, dx = -\int_{\mathbb{R}^N} (g(y)+g_\varepsilon(y))\overline{\varphi(y)}\, dy, \tag{3.7}$$

where we have used that

$$(u_x(x)-u_x(y))\overline{\varphi_x(y)} = -(u_y(y)-u_y(x))\overline{\varphi(y)}.$$

Combining (3.5), (3.6) and (3.7), and letting $\varepsilon \to 0$, we obtain

$$\mathrm{Re}\left[\int_{\mathbb{R}^N} (M([u]_{s,p}^A)(-\Delta_p^A)^s u + V(x)|u|^{p-2}u - |u|^{p_s^*-2}u)\overline{\varphi}\, dy\right] = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$. This ends the proof of the Theorem. ■

4. Concentration compactness principle

In this section we prove the concentration compactness principle Theorem 1.1. We follow some ideas from [28] and [24], adapted to the nonlocal framework.

We start with a technical lemma which constitutes a compactness result for complex functions in our setting. The proof follows analogously to [19, Lemma 2.4].

Lemma 4.1. *Let $0 < s < 1 < p$ be such that $sp < N$ and let $p \leq q < p_s^*$. Let $w \in L^\infty(\mathbb{R}^N)$ be such that there exist $\alpha > 0$ and $C > 0$ such that*

$$0 \leq w(x) \leq C|x|^{-\alpha}, \text{ for all } x \neq 0.$$

Then, if $\alpha > sq - N\frac{q-p}{p}$, we have $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N) \subset L^q_w(\mathbb{R}^N)$ compactly. That is, for any bounded sequence $\{u_k\} \subset \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, there exist a subsequence $\{u_{k_j}\} \subset \{u_k\}$ and a function $u \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ such that $u_{k_j} \rightharpoonup u$ weakly in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |u_{k_j}(x) - u(x)|^q \, w(x)dx \to 0 \quad \text{as } j \to \infty. \tag{4.1}$$

Proof. Let $\{u_k\} \subset \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ be a bounded sequence. Then, $\{u_k\}$ is also bounded in $L^p(\mathbb{R}^N)$. From the reflexivity of $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ and Theorem 3.5 in [21] for the particular case when the Orlicz function is a power, it follows that there exists $u \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ and a subsequence (that we still denote by $\{u_k\}$) such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \mathcal{D}^{s,p}_{A}(\mathbb{R}^N) \\ u_k &\to u \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^N). \end{aligned} \tag{4.2}$$

Let $R > 0$ to be chosen later. We write

$$\int_{\mathbb{R}^N} |u_k(x) - u(x)|^q \, w(x)dx = \left(\int_{|x|<R} + \int_{|x|\geq R}\right) |u_k(x) - u(x)|^q \, w(x)dx = I + II.$$

By Hölder's inequality and Sobolev Poincaré inequality, we can bound II as follows

$$
\begin{aligned}
II &\le C\left(\|u_k\|_{p_s^*}^q + \|u\|_{p_s^*}^q\right)\left(\int_{|x|\ge R} w^{\left(\frac{p_s^*}{q}\right)'}\,dx\right)^{\frac{1}{\left(\frac{p_s^*}{q}\right)'}} \\
&\le C\left(\int_{|x|\ge R} w^{\left(\frac{p_s^*}{q}\right)'}\,dx\right)^{\frac{1}{\left(\frac{p_s^*}{q}\right)'}}.
\end{aligned}
$$

Hence, using the hypothesis on w, we obtain that II goes to zero (uniformly in k) as R goes to infinity. So, given $\varepsilon > 0$ we choose $R > 0$ such that $II < \varepsilon$ for any k.

Finally, using the L^∞ bound on w and (4.2), we can bound I to get

$$
I \le \|w\|_\infty \|u_k - u\|_{q;B_R}^q \to 0 \quad \text{as } k \to \infty.
$$

Combining all these estimates, we have

$$
\limsup_{k\to\infty} \int_{\mathbb{R}^N} |u_k(x) - u(x)|^q\, w(x)dx \le \varepsilon,
$$

for every $\varepsilon > 0$. This finishes the proof. ■

Proof of Theorem 1.1. Since u_k is bounded in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, we have that $u_k \rightharpoonup u$, and that $|u_k|^{p_s^*}$ and $|D_s^A u_k|^p$ are bounded in $L^1(\mathbb{R}^N)$. Hence, there are Radon nonnegative measures ν and μ such that

$$
|u_k|^{p_s^*}\,dx \rightharpoonup \nu \quad \text{and} \quad |D_s^A u_k|^p\,dx \rightharpoonup \mu.
$$

Write

$$
\mu = \mu_f + \sum_{j\in J} \mu_j \delta_{x_j},
$$

where μ_f is a measure free of atoms. Since $\mu \ge 0$, then so is μ_f and hence

$$
\mu \ge \sum_{j\in J} \mu_j \delta_{x_j}. \tag{4.3}
$$

Moreover, we may assume

$$\mu_j = \mu(\{x_j\}) > 0, \text{ for all } j \in J.$$

Also, by the weak convergence,

$$\mu(\mathbb{R}^N) \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} |D_s^A u_k|^p \, dx < \infty,$$

which implies that J is at most countable. Indeed, we have that the series

$$\sum_{j\in J} \mu_j$$

converges in the sense of nets (that is, $\left\{\sum_{j\in F} \mu_j, \subset\right\}$ with $F \subset J$ finite, converges as a net). For each positive integer n let

$$S_n := \left\{ j \in J : \mu_j > \frac{1}{n} \right\}.$$

Suppose that S_n is infinite. Then for a countable set $S_n' \subset S_n$, $\sum_{j\in S_n'} \mu_j$ also converges (in the above sense) and so the series

$$\sum_{j\in S_n'} \mu_j$$

converges (in the usual sense of series). But this contradicts the definition of S_n. Thus, S_n is finite and since

$$J = \bigcup_n S_n$$

we conclude that J is at most countable.

Next, suppose that $u = 0$. We will obtain first the following reverse Hölder inequality:

$$S_A^{1/p} \left(\int_{\mathbb{R}^N} |\phi|^{p_s^*} \, d\nu \right)^{1/p_s^*} \leq \left(\int_{\mathbb{R}^N} |\phi|^p \, d\mu \right)^{1/p}, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N). \tag{4.4}$$

Indeed, fix $\phi \in C_0^\infty(\mathbb{R}^N)$. We have by (2.5) that

$$S_A \|\phi u_k\|_{p_s^*}^p \leq [\phi u_k]_{s,p}^A.$$

Next, by adding and subtracting the term

$$\frac{e^{i(x-y)A(\frac{x+y}{2})}\phi(x)u_k(y)}{|x-y|^{N+sp}}$$

in $D_s^A(\phi u_k)$ (recall (2.3)), we obtain that for any $\theta > 0$, there exists $C_\theta > 0$ such that

$$\begin{aligned}
|D_s^A(\phi u_k)(x)|^p \leq & \int_{\mathbb{R}^N} |\phi(x)u_k(x) - e^{i(x-y)A(\frac{x+y}{2})}\phi(x)u_k(y) \\
& + e^{i(x-y)A(\frac{x+y}{2})}\phi(x)u_k(y) - e^{i(x-y)A(\frac{x+y}{2})}\phi(y)u_k(y)|^p \frac{dy}{|x-y|^{N+sp}} \\
& \leq (1+\theta)\int_{\mathbb{R}^N} |\phi(x)|^p \frac{|u_k(x) - e^{i(x-y)A(\frac{x+y}{2})}u_k(y)|^p}{|x-y|^{N+sp}}\, dy \\
& + C_\theta \int_{\mathbb{R}^N} |u_k(y)|^p \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\, dy.
\end{aligned}$$

As a result,

$$[\phi u_k]_{s,p}^A = \int_{\mathbb{R}^N} |D_s^A(\phi u_k)(x)|^p\, dx \leq (1+\theta)\int_{\mathbb{R}^N} |\phi(x)|^p |D_s^A u_k(x)|^p\, dx + C_\theta \int_{\mathbb{R}^N} |u_k(y)|^p |D_s\phi(y)|^p\, dy. \tag{4.5}$$

Now, by Lemma 2.2 in [19], the weight $w = |D_s\phi|^p$ satisfies the assumptions of Lemma 4.1 with $p = q$, and so $u_k \to 0$ in $L_w^p(\mathbb{R}^N)$. Therefore,

$$\begin{aligned}
S_A\left(\int_{\mathbb{R}^N} |\phi|^{p_s^*}\, d\nu\right)^{p/p_s^*} & = \limsup_{n\to\infty}\left(S_A\|\phi u_k\|_{p_s^*}^p\right) \\
& \leq \limsup_{k\to\infty}\, [\phi u_k]_{s,p}^A \\
& = \limsup_{k\to\infty} \int_{\mathbb{R}^N} |D_s^A(\phi u_k)|^p(x)\, dx \qquad (4.6)\\
& \leq (1+\theta)\limsup_{k\to\infty}\int_{\mathbb{R}^N} |\phi|^p |D_s^A u_k|^p\, dx \quad \text{(by (4.5))} \\
& \leq (1+\theta)\int_{\mathbb{R}^N} |\phi|^p\, d\mu.
\end{aligned}$$

Sending $\theta \to 0$ in the above inequalities, we get (4.4).

Next, by approximation with smooth functions, we obtain for each Borel set $A \subset \mathbb{R}^N$,

$$S_A^{1/p}\nu(A)^{1/p_s^*} \leq \mu(A)^{1/p}. \tag{4.7}$$

Hence, ν is absolutely continuous with respect to μ and so, by Radon-Nykodym Theorem we may write

$$\nu(A) = \int_A D_\mu \nu \, d\mu,$$

where the density is, for μ-a.e. x,

$$D_\mu \nu(x) = \lim_{\varepsilon \to 0} \frac{\nu(B(x,\varepsilon))}{\mu(B(x,\varepsilon))}.$$

By (4.7),

$$\frac{\nu(B(x,\varepsilon))}{\mu(B(x,\varepsilon))} \leq \frac{1}{S_A^{p_s^*/p}} \frac{\mu(B(x,\varepsilon))^{p_s^*/p}}{\mu(B(x,\varepsilon))} = \frac{1}{S_A^{p_s^*/p}} \mu(B(x,\varepsilon))^{p_s^*/p-1} \to 0,$$

as $\varepsilon \to 0$ for μ-a.e. x in $\mathbb{R}^N \setminus \{x_j\}_{j\in J}$. Thus,

$$D_\mu \nu = 0$$

μ-a.e. in $\mathbb{R}^N \setminus \{x_j\}_{j\in J}$. Let now

$$\nu_j := D_\mu \nu(x_j)\mu_j.$$

Then, for any borel set A

$$\nu(A) = \int_A D_\mu \nu \, d\mu = \int_{\bigcup x_j \cap A} D_\mu \nu \, d\mu = \sum_{j\in J} \nu_j \delta_{x_j}(A). \tag{4.8}$$

By (4.7),

$$S_A \nu(B(x_j,\varepsilon))^{p/p_s^*} \leq \mu(B(x_j,\varepsilon)).$$

Sending $\varepsilon \to 0$ gives

$$S_A \nu_j^{p/p_s^*} \leq \mu_j. \tag{4.9}$$

Combining (4.3), (4.8) and (4.9), we prove the theorem when $u = 0$. Next, suppose $u \neq 0$. Then, define

$$v_k = u_k - u.$$

Thus, $v_k \rightharpoonup 0$ and moreover

$$\|D_s^A v_k\|_p,\ \|v_k\|_{p_s^*}$$

are bounded. So, there are measures $\hat{\mu}$ and $\hat{\nu}$ so that

$$|D_s^A v_k|^p\,dx \rightharpoonup \hat{\mu} \quad \text{and} \quad |v_k|^{p_s^*}\,dx \rightharpoonup \hat{\nu},$$

$$\hat{\mu} \geq \sum_j \mu_j \delta_{x_j},\ \ \hat{\nu} = \sum_j \nu_j \delta_{x_j}.$$

By the Brezis-Lieb Lemma [12, Theorem 1], for all $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \phi \left(|u_k|^{p_s^*} - |u_k - u|^{p_s^*} - |u|^{p_s^*} \right) dx = 0. \tag{4.10}$$

Since

$$\int_{\mathbb{R}^N} \phi |u_k|^{p_s^*}\,dx \to \int_{\mathbb{R}^N} \phi\,d\nu,$$
$$\int_{\mathbb{R}^N} \phi |u_k - u|^{p_s^*}\,dx \to \int_{\mathbb{R}^N} \phi\,d\hat{\nu},$$

it follows from (4.10) that

$$\int_{\mathbb{R}^N} \phi\,d\nu = \int_{\mathbb{R}^N} \phi\,d\hat{\nu} + \int_{\mathbb{R}^N} \phi |u|^{p_s^*}\,dx.$$

Since the above equality holds for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\nu = |u|^{p_s^*}\,dx + \sum_j \nu_j \delta_{x_j}.$$

For the measure μ,

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi |D_s^A u_k|^p\,dx - \int_{\mathbb{R}^N} \phi |D_s^A v_k|^p\,dx \\ &\quad = \int_{\mathbb{R}^N} \phi \int_0^1 p |D_s^A u_k - t D_s^A u|^{p-2} \mathrm{Re}[(D_s^A u_k - t D_s^A u)\overline{D_s^A u}]\,dt\,dx. \end{aligned} \tag{4.11}$$

Since $D_s^A u_k$ is bounded in $L^p(\mathbb{R}^N)$, we derive by (4.11) that there is $w \in L^{p'}(\mathbb{R}^N)$ such that

$$\lim_{k\to\infty}\left[\int_{\mathbb{R}^N}\phi|D_s^A u_k|^p\,dx-\int_{\mathbb{R}^N}\phi|D_s^A v_k|^p\,dx\right]=\int_{\mathbb{R}^N}\phi\ \mathrm{Re}[\,w\,\overline{D_s^A u}]\,dx.$$

On the other hand,

$$\lim_{k\to\infty}\left[\int_{\mathbb{R}^N}\phi|D_s^A u_k|^p\,dx-\int_{\mathbb{R}^N}\phi|D_s^A v_k|^p\,dx\right]=\int_{\mathbb{R}^N}\phi\,d\mu-\int_{\mathbb{R}^N}\phi\,d\hat{\mu}.$$

As a result,

$$\int_{\mathbb{R}^N}\phi\,d\mu-\int_{\mathbb{R}^N}\phi\,d\hat{\mu}=\int_{\mathbb{R}^N}\phi\ \mathrm{Re}[\,w\,\overline{D_s^A u}]\,dx$$

which implies that μ and $\hat{\mu}$ have the same atoms:

$$\sum_j \mu_j\delta_{x_j}.$$

Finally, since for each compact set K, $u_k \rightharpoonup u$ in $\mathcal{D}_A^{s,p}(K)$ and $|D_s^A u_k|\,dx \rightharpoonup \mu$, there holds

$$\int_K |D_s^A u|^p\,dx \le \liminf_{k\to\infty}\int_K |D_s^A u_k|^p\,dx \le \limsup_{k\to\infty}\int_K |D_s^A u_k|^p\,dx \le \mu(K). \tag{4.12}$$

Thus, if $A\subset \mathbb{R}^N\backslash \bigcup\{x_j\}$, is borel and bounded, then there is a sequence of compact sets K_j satisfying $K_j \nearrow A$.

So, by (4.12) and monotone convergence Theorem,

$$\mu_f(A)=\lim_{j\to\infty}\mu_f(K_j)\ge \lim_{j\to\infty}\int_{K_j}|D_A^s u|^p\,dx=\int_A |D_A^s u|^p\,dx.$$

Therefore,

$$\mu_f \ge |D_s^A u|^p\,dx.$$

Hence, we conclude

$$\mu \ge |D_s^A u|^p\,dx+\sum_j \mu_j\delta_{x_j}.$$

As we wanted to prove.

■

5. Mountain pass geometry of the energy functional

We recall the energy functional associated to the given Equation (1.1):

$$I(u) = \frac{1}{p}\mathcal{M}\left([u]_{s,p}^{A}\right) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\,dx, \quad u \in \mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N).$$

The following mountain pass Theorem without the Palais-Smale condition will be used to get critical point of I. Its proof can be found in [7, pag. 272].

Theorem 5.1. *Let E be a Banach space and $I \in C^1(E,\mathbb{R})$. Suppose that there exist a neighbourhood U of* 0 *in E and a constant α satisfying the following conditions*

(i) $I(u) \geq \alpha$ *for all* $u \in \partial U$,

(ii) $I(0) < \alpha$,

(iii) *there exists* $u_0 \notin U$ *satisfying* $I(u_0) < \alpha$.

Let

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0,\, \gamma(1) = u_0\}$$

and

$$c = \inf_{\gamma\in\Gamma}\max_{u\in\gamma([0,1])} I(u) \geq \alpha.$$

Then, there exists a sequence $u_k \in E$ such that

$$I(u_k) \to c \quad \text{and} \quad I'(u_k) \to 0 \text{ in } E',$$

as $k \to \infty$.

A sequence satisfying the conclusion of Theorem 5.1 is called a Palais-Smale sequence.

In the next lemmas we will prove that I satisfies the geometric conditions of Theorem 5.1.

Lemma 5.2. *We have that* $I \in C^1(\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N), \mathbb{R})$.

Proof. Let $u \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, we split

$$I(u) = \frac{1}{p}\mathcal{M}\left([u]^A_{s,p}\right) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N}|u|^{p_s^*}\,dx = I_1(u) + I_2(u) + I_3(u).$$

It is enough to prove that $I_1, I_2, I_3 \in C^1(\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N), \mathbb{R})$. We start computing the Gateux derivatives

$$\langle I_1'(u), \varphi\rangle = \frac{1}{p}\frac{d}{dt}\mathcal{M}\left([u+t\varphi]^A_{s,p}\right)\big|_{t=0} = Re\left[M([u]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u|^{p-2}D_{s,A}u\overline{D_{s,A}\varphi}\,d\eta\right],$$

$$\langle I_2'(u), \varphi\rangle = Re\left[\int_{\mathbb{R}^N}|u|^{p-2}u\overline{\varphi}V(x)\,dx\right],$$

and

$$\langle I_3'(u), \varphi\rangle = Re\left[\int_{\mathbb{R}^N}|u|^{p_s^*-2}u\overline{\varphi}\,dx\right].$$

We first prove that $I_2 \in C^1(\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N), \mathbb{R})$. In fact, suppose that $u_k \to u$ in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, then $u_k \to u$ in $L^p_V(\mathbb{R}^N)$, by [[11], Theorem 4.9], there exists a subsequence that we will still denote u_k such that $u_k \to u$ a.e. and $h \in L^p_V(\mathbb{R}^N)$ such that $|u_k| \le h$. Then,

$$\begin{aligned}|\langle I_2'(u_k) - I_2'(u), \varphi\rangle| &= \left|Re\left[\int_{\mathbb{R}^N}(|u_k|^{p-2}u_k - |u|^{p-2}u)\overline{\varphi}V(x)\,dx\right]\right| \\ &\le \||u_k|^{p-2}u_k - |u|^{p-2}u\|_{p',V}\|\varphi\|_{p,V}.\end{aligned}$$

As $|u_k|^{p-2}u_k - |u|^{p-2}u \to 0$ a.e and $||u_k|^{p-2}u_k - |u|^{p-2}u|^{\frac{p}{p-1}} \le |h|^p + |u|^p$, by Dominated convergence Theorem, it follows that $I_2'(u_k) \to I_2'(u)$ as k goes to infinite. The proof of $I_3 \in C^1(\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N), \mathbb{R})$ follows analogously, noting that by (2.5) if $u_k \to u$ in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, then $u_k \to u$ in $L^{p_s^*}(\mathbb{R}^N)$. Finally, to prove that $I_1 \in C^1(\mathcal{D}^{s,p}_{A,V}, \mathbb{R})(\mathbb{R}^N)$, observe that if $u_k \to u$ in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$ so $u_k \to u$ in $\mathcal{D}^{s,p}_{A}(\mathbb{R}^N)$ and the proof follows analogously.

∎

Lemma 5.3. *The functional I satisfies the geometric conditions $(i)-(iii)$ from Theorem 5.1.*

Proof. First, observe that for $\alpha > 0$, $I(0) = 0 < \alpha$. Hence, I verifies (ii) trivially for any $\alpha > 0$.

To prove that I satisfies (i), let $U = B(0,\rho)$, with $0 < \rho < 1$ to be chosen. If $u \in \partial B(0,\rho)$, then

$$[u]^A_{s,p} + \int_{\mathbb{R}^N} V(x)|u|^p\,dx = \rho^p.$$

Hence,

$$\begin{aligned}
I(u) &= \frac{1}{p}\mathcal{M}([u]^A_{s,p}) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\,dx - \frac{1}{p^*_s}\|u\|^{p^*_s}_{p^*_s} \\
&\geq \frac{\mathcal{M}(1)}{p}([u]^A_{s,p})^\theta + \min\{\mathcal{M}(1),1\}\left(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^\theta - \frac{1}{S_A p^*_s}\rho^{p^*_s} \quad (\text{by } (\mathrm{HM})_2,\ \theta > 1) \\
&\geq \frac{\min\{\mathcal{M}(1),1\}}{p}\left(([u]^A_{s,p})^\theta + \left(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^\theta\right) - \frac{1}{S_A p^*_s}\rho^{p^*_s} \\
&\geq \frac{\min\{\mathcal{M}(1),1\}}{p}c_\theta \rho^{p\theta} - \frac{1}{S_A p^*_s}\rho^{p^*_s} \\
&\geq \alpha,
\end{aligned}$$

for some $\alpha > 0$, choosing ρ small enough and recalling $\theta < p^*_s/p$ by $(HM)_2$.

Finally, we prove that I satisfies (iii). Let u be fixed such that $[u]^A_{s,p} = 1$ and $\|u\|_{p,V} > 0$. For $t > 0$, we consider tu. Then,

$$\begin{aligned}
I(tu) &= \frac{1}{p}\mathcal{M}([tu]^A_{s,p}) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|tu|^p\,dx - \frac{1}{p^*_s}\int_{\mathbb{R}^N} V(x)|tu|^{p^*_s}\,dx \\
&\leq \mathcal{M}(1)([tu]^A_{s,p})^\theta + t^p\|u\|^p_{p,V} - \frac{1}{p^*_s}t^{p^*_s}\|u\|^{p^*_s}_{p^*_s,V} \\
&\leq \mathcal{M}(1)t^{p\theta} + t^p\|u\|^p_{p,V} - \frac{1}{p^*_s}t^{p^*_s}\|u\|^{p^*_s}_{p^*_s,V}.
\end{aligned}$$

As $\frac{p^*_s}{p} > \theta$, taking limit as t goes to infinity, we can assure that $I(tu) < 0$. This completes the proof. ■

By Lemma 5.2 and Lemma 5.3, there is a sequence $u_k \in D^{s,p}_{A,V}(\mathbb{R}^N)$ such that

$$I(u_k) \to c \quad \text{and} \qquad I'(u_k) \to 0 \text{ in } (D^{s,p}_{A,V}(\mathbb{R}^N))',$$

as $k \to \infty$. In the next section we will study convergence properties of this sequence.

6. Existence of solutions to (1.1)

In this section we will prove that the limit of the Palais-Smale sequence u_k founded in the previous section is actually a solution of (1.1). We will prove that in a sequence of steps.

Lemma 6.1. *The Palais-Smale sequence $\{u_k\}$ is bounded in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$.*

Proof. First, we consider the case $a = \inf[u_k]^A_{s,p} > 0$. Using $(HM)_1$ we can bound $I(u_k) - \frac{1}{p^*_s}\langle I'(u_k), u_k\rangle$ from below. In fact,

$$\begin{aligned} I(u_k) - \frac{1}{p^*_s}\langle I'(u_k), u_k\rangle &\geq \frac{1}{p}\mathcal{M}([u_k]^A_{s,p}) - \frac{1}{p^*_s}M([u_k]^A_{s,p})([u_k]^A_{s,p}) + \left(\frac{1}{p} - \frac{1}{p^*_s}\right)\int_{\mathbb{R}^N} V(x)|u_k|^p\,dx \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{p^*_s}\right)\delta[u_k]^A_{s,p} + \left(\frac{1}{p} - \frac{1}{p^*_s}\right)\int_{\mathbb{R}^N} V(x)|u_k|^p\,dx \\ &\geq \min\left\{\left(\frac{1}{\theta p} - \frac{1}{p^*_s}\right)\delta, \left(\frac{1}{p} - \frac{1}{p^*_s}\right)\right\}\|u_k\|^p, \quad \text{with } \delta = \delta(d). \end{aligned}$$

On the other hand,

$$\begin{aligned} I(u_k) - \frac{1}{p^*_s}\langle I'(u_k), u_k\rangle &\leq I(u_k) + \frac{1}{p^*_s}\|I'(u_k)\|\|u_k\| \\ &\leq c_1 + c_2\|u_k\|. \end{aligned}$$

Thus,

$$C\|u_k\|^p \leq c_1 + c_2\|u_k\|.$$

and it is easy to see that u_k is bounded.

Suppose now that $a = 0$. If 0 is an isolated point, then we can build a subsequence that we still denote u_k such that $a = \inf[u_k]^A_{s,p} > 0$ and we proceed as before.

If 0 is not an isolated point, we have, up to a subsequence, that $[u_k]^A_{s,p} \to 0$ and we can use $(HM)_3$, so as before we obtain the following bound from below

$$I(u_k) - \frac{1}{p_s^*}\langle I'(u_k), u_k\rangle \geq c_0 \min\left\{\left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right), \left(\frac{1}{p} - \frac{1}{p_s^*}\right)\right\}\left\{o(1) + \int_{\mathbb{R}^N} V(x)|u_k|^p\, dx\right\}. \tag{6.1}$$

Moreover,

$$I(u_k) - \frac{1}{p_s^*}\langle I'(u_k), u_k\rangle \leq c_1 + o(1) + c_2\left(\int_{\mathbb{R}^N} V(x)|u_k|^p\, dx\right)^{\frac{1}{p}}. \tag{6.2}$$

By (6.1) and (6.2) follows that if $\int_{\mathbb{R}^N} V(x)|u_k|^p\, dx \to \infty$, we have a contradiction. So, $\int_{\mathbb{R}^N} V(x)|u_k|^p\, dx$ is bounded and consequently $\{u_k\}$ is bounded as well. The proof is completed.

■

Remark 6.2. Observe that since u_k is bounded in $\mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N)$, the sequence of real numbers $[u_k]^A_{s,p}$ is bounded and hence, up to a subsequence not relabel,

$$[u_k]^A_{s,p} \to d, \text{for some scalar } d \geq 0.$$

However, $d > 0$. Indeed, if $d = 0$, then by (2.5), $\|u_k\|_{p_s^*} \to 0$. Hence, since $I'(u_k) \to 0$ as $k \to \infty$, we obtain

$$\big\langle I'(u_k), u_k\big\rangle \to 0.$$

This implies that

$$\int_{\mathbb{R}^N} |u_k|^p V(x)\, dx \to 0 \quad \text{as } k \to \infty.$$

It then follows that

$$I(u_k) \to 0,$$

which contradicts the fact that the critical level c from Theorem 5.1 is positive.

Therefore, in what follows, we will assume that, up to a subsequence that we do not relabel,

$$[u_k]_{s,p}^A \to d > 0.$$

Since u_k is a bounded sequence in $\mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N)$, we can apply Theorem 1.1 to derive the existence of a countable set $\{x_j\}_{j\in J}$ and measures μ and ν so that the conclusions of Theorem 1.1 hold. By the properties of the sequence u_k we derive the following result.

Lemma 6.3. *The set $\{x_j\}_{j\in J}$ is finite.*

Proof. Let x_j be fixed and take $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \phi \le 1$ and

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \le 1 \\ 0, & \text{if } |x| \ge 2. \end{cases}$$

For $\varepsilon > 0$, define

$$\phi_\varepsilon(x) := \phi\left(\frac{x - x_j}{\varepsilon}\right).$$

We first prove that $\phi_\varepsilon u_k$ is bounded (with respect to k) in $\mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N)$. Clearly,

$$\|\phi_\varepsilon u_k\|_{p,V} \le \|u_k\|_{p,V} \le C.$$

Next, by (4.5),

$$\begin{aligned} [\phi_\varepsilon u_k]_{s,p}^A &\le C\bigg(\int_{\mathbb{R}^N} |\phi_\varepsilon(y)|^p |D_s^A u_k(y)|^p + |u_k(y)|^p |D^s \phi_\varepsilon(y)|^p \, dy\bigg) \\ &\le C\bigg([u_k]_{s,p}^A + \int_{\mathbb{R}^N} |u_k(y)|^p |D^s \phi_\varepsilon(y)|^p \, dy\bigg). \end{aligned} \tag{6.3}$$

Observe that $[u_k]_{s,p}^A$ is bounded. Regarding the term

$$\int_{\mathbb{R}^N} |u_k(y)|^p |D^s\phi_\varepsilon(y)|^p \, dy, \tag{6.4}$$

we have by Corollary 2.3 in [19] that

$$|D^s\phi_\varepsilon(x)|^p \leq C \min\left\{\varepsilon^{-sp}, \varepsilon^N |x-x_j|^{-(N+sp)}\right\}.$$

Therefore, $|D^s\phi_\varepsilon(x)|^p$ is bounded. Moreover, since $|u_k|$ is bounded in L_V^p, so is in L^p and hence (6.4) is also bounded.

Next, recall that

$$[u_k]_{s,p}^A \to d > 0.$$

Then,

$$M([u_k]_{s,p}^A) \to M(d) > 0.$$

Since $I'(u_k) \to 0$, we may write

$$\begin{aligned} &Re\left[M([u_k]_{s,p}^A)\int_{\mathbb{R}^{2N}} |D_{s,A}u_k(x,y)|^{p-2} D_{s,A}u_k(x,y)\overline{D_{s,A}(\phi_\varepsilon u_k(x,y))}\, d\eta\right] \\ &= -\int_{\mathbb{R}^N} |u_k|^{p-2}u_k\phi_\varepsilon \bar{u_k} V(x)\, dx + \int_{\mathbb{R}^N} |u_k|^{p_s^*-2}u_k\phi_\varepsilon\bar{u_k}\, dx + o(1) \\ &= -\int_{\mathbb{R}^N} |u_k|^p \phi_\varepsilon V(x)\, dx + \int_{\mathbb{R}^N} |u_k|^{p_s^*}\phi_\varepsilon\, dx + o(1) \\ &\leq \int_{\mathbb{R}^N} |u_k|^{p_s^*}\phi_\varepsilon\, dx + o(1). \end{aligned} \tag{6.5}$$

We write the first term in (6.5) as

$$\begin{aligned} &M([u_k]_{s,p}^A)\, Re\bigg[\int_{\mathbb{R}^{2N}} |D_{s,A}u_k(x,y)|^p \phi_\varepsilon(x)\, d\eta \\ &+ \int_{\mathbb{R}^{2N}} |D_{s,A}u_k(x,y)|^{p-2} D_{s,A}u_k(x,y) D_s\phi_\varepsilon(x,y)\overline{e^{i(x-y)A(\frac{x+y}{2})}u_k(y)}\, d\eta\bigg]. \end{aligned} \tag{6.6}$$

We consider now the second term in (6.6). Since

$$|D_{s,A}u_k(x,y)|^{p-2}D_{s,A}u_k(x,y)$$

is bounded in $L^{p'}_\eta(\mathbb{R}^{2N})$, there is $w_1 = w_1(x,y) \in L^{p'}_\eta(\mathbb{R}^{2N})$ such that, up to a subsequence

$$|D_{s,A}u_k(x,y)|^{p-2}D_{s,A}u_k(x,y) \rightharpoonup w_1 \quad \text{in } L^{p'}_\eta(\mathbb{R}^{2N}). \tag{6.7}$$

We next prove that

$$D_s\phi_\varepsilon(x,y)\overline{e^{i(x-y)A(\frac{x+y}{2})}u_k(y)}$$

converges strongly in $L^p_\eta(\mathbb{R}^{2N})$ as $k \to \infty$. Indeed, since

$$\|D_s\phi_\varepsilon(x,y)e^{i(x-y)A(\frac{x+y}{2})}(u_k-u)\|^p_{\eta,p} = \int_{\mathbb{R}^N}|u_k(y)-u(y)|^p|D^s\phi_\varepsilon(y)|^p\,dy,$$

we obtain from Lemma 4.1 with

$$w = |D^s\phi_\varepsilon(y)|^p \in L^\infty(\mathbb{R}^N),$$

that, up to a subsequence that we do not re label,

$$\int_{\mathbb{R}^N}|u_k(y)-u(y)|^p|D^s\phi_\varepsilon(y)|^p\,dy \to 0. \tag{6.8}$$

Combining the weak convergence (6.7) with the strong convergence (6.8), we obtain that

$$M([u_k]^A_{s,p})\,Re\Big[\int_{\mathbb{R}^{2N}}|D_{s,A}u_k(x,y)|^{p-2}D_{s,A}u_k(x,y)D_s\phi_\varepsilon(x,y)\overline{e^{i(x-y)A(\frac{x+y}{2})}u_k(y)}\,d\eta\Big)$$
$$\to M(d)Re\Big[\int_{\mathbb{R}^{2N}}w_1(x,y)e^{-i(x-y)A(\frac{x+y}{2})}D_s\phi_\varepsilon(x,y)\overline{u(y)}\,d\eta\Big].$$

In this way, letting $k \to \infty$ in (6.5) and taking (6.6) into account, we obtain

$$M(d)\int_{\mathbb{R}^N}\phi_\varepsilon(x)\,d\mu + M(d)Re\Big[\int_{\mathbb{R}^{2N}}w_1(x,y)e^{-i(x-y)A(\frac{x+y}{2})}D_s\phi_\varepsilon(x,y)\overline{u(y)}\,d\eta\Big] \le \int_{\mathbb{R}^N}\phi_\varepsilon(x)\,d\nu. \tag{6.9}$$

Next, we will show that

$$\int_{\mathbb{R}^{2N}} w_1 e^{-i(x-y)A(\frac{x+y}{2})} D_s\phi_\varepsilon(x,y)\overline{u(y)}\, d\eta \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.10}$$

Note that

$$\int_{\mathbb{R}^N} ||u_k|^{p-2}u_k|^{p/p-1} V(x)\, dx = \int_{\mathbb{R}^N} |u_k|^p V(x)\, dx \le C$$

and

$$\int_{\mathbb{R}^N} ||u_k|^{p_s^*-2}u_k|^{p_s^*/p_s^*-1}\, dx = \int_{\mathbb{R}^N} |u_k|^{p_s^*}\, dx \le C.$$

Hence, there are $w_2 \in L_V^{p'}(\mathbb{R}^N)$ and $w_3 \in L^{(p_s^*)'}(\mathbb{R}^N)$ such that

$$|u_k|^{p-2}u_k \rightharpoonup w_2 \quad \text{in } L_V^{p'}(\mathbb{R}^N), \tag{6.11}$$

and

$$|u_k|^{p_s^*-2}u_k \rightharpoonup w_3 \quad \text{in } L^{(p_s^*)'}(\mathbb{R}^N). \tag{6.12}$$

Since $\langle I'(u_k), v\rangle \to 0$ for any $v \in \mathcal{D}_{A,V}^{s,p}(\mathbb{R}^N)$, we get

$$Re\Big[M([u_k]_{s,p}^A) \int_{\mathbb{R}^{2N}} |D_{s,A}u_k(x,y)|^{p-2} D_{s,A}u_k(x,y)\overline{D_{s,A}v(x,y)}\, d\eta$$
$$+ \int_{\mathbb{R}^N} |u_k|^{p-2}u_k\overline{v}V(x)\, dx - \int_{\mathbb{R}^N} |u_k|^{p_s^*-2}u_k\overline{v}\, dx\Big] \to 0,$$

and so by (6.7), (6.11) and (6.12),

$$Re\Big[M(d) \int_{\mathbb{R}^{2N}} w_1\overline{(D_{s,A}v(x,y))}\, d\eta + \int_{\mathbb{R}^N} w_2\overline{v}V(x)\, dx - \int_{\mathbb{R}^N} w_3\overline{v}\, dx\Big] = 0. \tag{6.13}$$

Taking $v = u\phi_\varepsilon$ in (6.13), we deduce

$$Re\Big[M(d)\int_{\mathbb{R}^{2N}} w_1\overline{D_{s,A}(u\phi_\varepsilon)}\, d\eta + \int_{\mathbb{R}^N} w_2\overline{u\phi_\varepsilon}V(x)\, dx - \int_{\mathbb{R}^N} w_3\overline{u\phi_\varepsilon}\, dx\Big] = 0,$$

and so

$$\begin{aligned} &Re\left[M(d)\int_{\mathbb{R}^{2N}} w_1 D_s\phi_\varepsilon(x,y)e^{-i(x-y)A(\frac{x+y}{2})}\overline{u}\,d\eta\right] \\ &\quad = Re\left[M(d)\int_{\mathbb{R}^{2N}} w_1\overline{D_{s,A}u(x,y)}\phi_\varepsilon\,d\eta - \int_{\mathbb{R}^N} w_2\overline{u}\phi_\varepsilon V(x)\,dx + \int_{\mathbb{R}^N} w_3\overline{u}\phi_\varepsilon\,dx\right]. \end{aligned} \tag{6.14}$$

By dominated convergence theorem and the facts that $w_1\overline{D_{s,A}u(x,y)} \in L^1(\mathbb{R}^{2N})$, $w_2\overline{u} \in L^1_V(\mathbb{R}^N)$ and $w_3\overline{u} \in L^1(\mathbb{R}^N)$, we obtain that the right-hand side of (6.14) tends to 0 as $\varepsilon \to 0$. This proves the claim (6.10).

Finally, taking $\varepsilon \to 0$ in (6.9), we obtain

$$M(d)\mu_j \le \nu_j.$$

By (1.7) it follows that

$$\nu_j \ge M(d)\mu_j \ge M(d)S_A\nu_j^{p/p_s^*}$$

and so

$$\nu_j \ge (M(d)S_A)^{\frac{1}{(1-p/p_s^*)}}.$$

Since $\sum_j \nu_j < \infty$, we conclude that J is finite.

■

We next prove that $u_k \to u$ in $L^{p_s^*}(K)$ for any $K \subset \mathbb{R}^N \setminus \bigcup x_j$.

Lemma 6.4. *For any compact $K \subset \mathbb{R}^N \setminus \bigcup x_j$, there holds that $u_k \to u$ in $L^{p_s^*}(K)$.*

Proof. Let $K \subset \mathbb{R}^N \setminus \bigcup x_j$ be compact. We let

$$\gamma := \text{dist }(K, \{x_j\}_j),$$

hence by Lemma 6.3, $\gamma > 0$. Let $R > 0$ such that $K \subset B_R(0)$ and for $\varepsilon > 0$ define

$$A_\varepsilon := \{x \in B_R(0) : \text{dist}(x, K) < \varepsilon\}, \quad 0 < \varepsilon < \gamma.$$

Finally, consider $\phi_\varepsilon \in C_0(\mathbb{R}^N)$ such that $\phi_\varepsilon \in [0,1]$ and

$$\phi_\varepsilon = \begin{cases} 1, & \text{if} \quad x \in A_{\varepsilon/2} \\ \\ 0, & \text{if} \quad x \in \mathbb{R}^N \setminus A_\varepsilon. \end{cases}$$

Observe that since

$$K \subset A_{\varepsilon/2} \subset A_\varepsilon \subset \mathbb{R}^N \setminus \bigcup x_j \cap B_R(0),$$

we obtain

$$\int_K |u_k|^{p_s^*}\,dx \leq \int_{A_\varepsilon} \phi_\varepsilon |u_k|^{p_s^*}\,dx = \int_{\mathbb{R}^N} \phi_\varepsilon |u_k|^{p_s^*}\,dx.$$

Hence, by (1.6),

$$\limsup_{k\to\infty} \int_K |u_k|^{p_s^*}\,dx \leq \int_{\mathbb{R}^N} \phi_\varepsilon\,d\nu = \int_{\mathbb{R}^N} \phi_\varepsilon |u|^{p_s^*}\,dx \leq \int_{A_\varepsilon} |u|^{p_s^*}\,dx. \tag{6.15}$$

By dominated convergence theorem, if $\varepsilon \to 0$ in (6.15), then

$$\limsup_{k\to\infty} \int_K |u_k|^{p_s^*}\,dx \leq \int_K |u|^{p_s^*}\,dx. \tag{6.16}$$

Next, by Fatou's Lemma

$$\int_K |u|^{p_s^*}\,dx \leq \liminf_{k\to\infty} \int_K |u_k|^{p_s^*}\,dx. \tag{6.17}$$

Hence, combining (6.16) and (6.17) we get

$$\lim_{k\to\infty} \int_K |u_k|^{p_s^*}\,dx = \int_K |u|^{p_s^*}\,dx. \tag{6.18}$$

Recalling that $u_k \to u$ a.e. by for instance [21, Theorem 3.5] and weakly in $L^{p_s^*}(\mathbb{R}^N)$, we get from the Brezis-Lieb Lemma that

$$\int_K \left(|u_k|^{p_s^*} - |u_k - u|^{p_s^*} - |u|^{p_s^*} \right)\,dx \to 0,$$

hence by (6.18), $u_k \to u$ in $L^{p_s^*}(K)$. This ends the proof of the lemma. ∎

Lemma 6.5. *For any compact $K \subset \mathbb{R}^N \setminus \bigcup x_j$, we have*

$$Re\left[\int_{K\times K}\left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\left(\overline{D_{s,A}u_k - D_{s,A}u}\right)d\eta\right] \to 0, \tag{6.19}$$

as $k \to \infty$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \varphi \le 1$, $\varphi = 1$ in K and

$$\text{supp}(\varphi) \cap \{x_j\}_{j\in J} = \emptyset.$$

First, observe that

$$\begin{aligned}
&\text{Re}\left[\left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\left(\overline{D_{s,A}u_k - D_{s,A}u}\right)\right]\\
&\quad= |D_{s,A}u_k|^p + |D_{s,A}u|^p - |D_{s,A}u_k|^{p-2}\text{Re}(D_{s,A}u_k\overline{D_{s,A}u}) - |D_{s,A}u|^{p-2}\text{Re}(D_{s,A}u\overline{D_{s,A}u_k})\\
&\quad\ge |D_{s,A}u_k|^p + |D_{s,A}u|^p - |D_{s,A}u_k|^{p-1}|D_{s,A}u| - |D_{s,A}u|^{p-1}|D_{s,A}u_k|\\
&\quad= |D_{s,A}u_k|^{p-1}(|D_{s,A}u_k| - |D_{s,A}u|) + |D_{s,A}u|^{p-1}(|D_{s,A}u| - |D_{s,A}u_k|)\\
&\quad= (|D_{s,A}u_k|^{p-1} - |D_{s,A}u|^{p-1})(|D_{s,A}u_k| - |D_{s,A}u|) \ge 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\le \text{Re}\left[\int_{K\times K}\left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\left(\overline{D_{s,A}u_k - D_{s,A}u}\right)d\eta\right]\\
&\le \text{Re}\left[\int_{\mathbb{R}^{2N}}\left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\left(\overline{D_{s,A}u_k - D_{s,A}u}\right)\varphi\, d\eta\right]\\
&= \text{Re}\left[\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_k(\overline{D_{s,A}u_k - D_{s,A}u})\,\varphi\, d\eta\right]\\
&- \text{Re}\left[\int_{\mathbb{R}^{2N}}|D_{s,A}u|^{p-2}D_{s,A}u(\overline{D_{s,A}u_k - D_{s,A}u})\,\varphi\, d\eta\right].
\end{aligned} \tag{6.20}$$

Since $(u_k - u)\varphi$ is bounded in $\mathcal{D}^{s,p}_{A,V}$ (see the beginning of the proof of Lemma 6.3), we have

$$\left\langle I'(u_k), (u_k - u)\varphi\right\rangle \to 0$$

and then

$$
\begin{aligned}
o(1) = \langle I'(u_k), (u_k - u)\varphi\rangle = \mathrm{Re}\Big[M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}} |D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}(u_k-u)\varphi}\,d\eta \\
+\int_{\mathbb{R}^N}|u_k|^{p-2}u_k\overline{(u_k-u)}\varphi V(x)\,dx - \int_{\mathbb{R}^N}|u_k|^{p_s^*-2}u_k\overline{(u_k-u)}\varphi\,dx\Big].
\end{aligned}
\tag{6.21}
$$

Observe that for $K' :=$ supp φ,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}|u_k|^{p-2}u_k\overline{(u_k-u)}\varphi V(x)\,dx + \int_{\mathbb{R}^N}|u_k|^{p_s^*-2}u_k\overline{(u_k-u)}\varphi\,dx\right| \\
&\quad\le \left|\int_{K'}|u_k|^{p-1}|u_k-u|V(x)\,dx + \int_{K'}|u_k|^{p_s^*-1}|u_k-u|\,dx\right| \\
&\quad\le \|V\|_{\infty,K'}\|u_k\|^{p-1}_{p,K'}\|u_k-u\|_{p,K'} + \|u_k\|^{p_s^*-1}_{p_s^*}\|u_k-u\|_{p_s^*,K'} \to 0,
\end{aligned}
$$

where the convergences at the end are consequences of [21, Theorem 3.5] and Lemma 6.4. Consequently, by (6.21), we get

$$
\mathrm{Re}\Big[M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}(u_k-u)\varphi}\,d\eta\Big] \to 0, \text{ as } k\to\infty.
$$

Next, we split the above term as

$$
\begin{aligned}
o(1) &= \mathrm{Re}\Big[M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}(u_k-u)\varphi}\,d\eta\Big] \\
&= \mathrm{Re}\Big[M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}(u_k-u)}\varphi\,d\eta\Big] \\
&\quad + \mathrm{Re}\Big[M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_kD_s\varphi\,\overline{e^{i(x-y)A(\frac{x+y}{2})}(u_k-u)}\,d\eta\Big].
\end{aligned}
\tag{6.22}
$$

Next, observe that by Hölder's inequality, the fact that $[u_k]^A_{s,p}$ is bounded and by Lemma 4.1,

$$
\begin{aligned}
&\left|M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_kD_s\varphi\,\overline{e^{i(x-y)A(\frac{x+y}{2})}(u_k-u)}\,d\eta\right| \\
&\quad\le C.\left([u_k]^A_{s,p}\right)^{p-1/p}\left(\int_{\mathbb{R}^{2N}}|D_s\varphi(x,y)|^p|u_k(x)-u(x)|^p\,d\eta\right)^{1/p} \\
&\quad\le C\int_{\mathbb{R}^N}|D^s\varphi(x)|^p|u_k(x)-u(x)|^p\,dx = o(1).
\end{aligned}
\tag{6.23}
$$

Hence, combining (6.22) and (6.23), we obtain that

$$\mathrm{Re}\left[M([u_k]_{s,p}^A)\int_{\mathbb{R}^{2N}}|D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}(u_k-u)}\varphi\,d\eta\right]=o(1). \tag{6.24}$$

Finally, by weak convergence,

$$\mathrm{Re}\left[\int_{\mathbb{R}^{2N}}|D_{s,A}u|^{p-2}D_{s,A}u(\overline{D_{s,A}u_k-D_{s,A}u})\,\varphi\,d\eta\right]=o(1). \tag{6.25}$$

Therefore, by (6.24) and (6.25), and recalling (6.20), we get the conclusion (6.19). ■

We next prove the almost everywhere convergence of the magnetic fractional gradients. We will use the following known inequalities for vectors $\xi,\eta\in\mathbb{R}^N$:

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)\geq C|\xi-\eta|^p,\quad p\geq 2 \tag{6.26}$$

and

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)\geq (p-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}|\xi-\eta|^p,\quad 1<p<2. \tag{6.27}$$

We also appeal to the following fact: for any $a,b\in\mathbb{C}$,

$$\mathrm{Re}\left[(|a|^{p-2}a-|b|^{p-2}b)\overline{(a-b)}\right]=(|a|^{p-2}a-|b|^{p-2}b)\cdot(a-b), \tag{6.28}$$

where in the right-hand side we have the scalar product between the vectors $a=(Re(a),\,Im(a))$ and $b=(Re(b),\,Im(b))$.

Lemma 6.6. *We have that*

$$D_{s,A}u_k(x,y)\to D_{s,A}u(x,y)\quad a.e.\ (x,y)\in\mathbb{R}^{2N}.$$

Proof. Let $K \subset \mathbb{R}^N \setminus \{x_j\}_{j\in J}$ be compact. Recalling Lemma 6.5, the inequalities (6.26) and (6.28), we have for $p \geq 2$,

$$\begin{aligned} o(1) &= \mathrm{Re}\left[\int_{K\times K} \left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\left(\overline{D_{s,A}u_k - D_{s,A}u}\right) d\eta\right] \\ &= \int_{K\times K} \left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\cdot\left(D_{s,A}u_k - D_{s,A}u\right) d\eta \\ &\geq C\int_{K\times K} |D_{s,A}u_k - D_{s,A}u|^p \, d\eta. \end{aligned}$$

Therefore, $D_{s,A}u_k \to D_{s,A}u$ in $L^p_\eta(K\times K)$ for any compact $K \subset \mathbb{R}^N \setminus \{x_j\}_{j\in J}$. Hence, the lemma holds in this case. For the singular case, we proceed as follows

$$\begin{aligned} \int_{K\times K} |D_{s,A}u_k - D_{s,A}u|^p \, d\eta &\leq \int_{K\times K} \frac{|D_{s,A}u_k - D_{s,A}u|^p}{(|D_{s,A}u_k| + |D_{s,A}u|)^{(2-p)p/2}}\left(|D_{s,A}u_k| + |D_{s,A}u|\right)^{(2-p)p/2} \, d\eta \\ &\leq \left\|\frac{|D_{s,A}u_k - D_{s,A}u|^p}{(|D_{s,A}u_k| + |D_{s,A}u|)^{(2-p)p/2}}\right\|_{2/p,\eta,K\times K}\left\|\left(|D_{s,A}u_k| + |D_{s,A}u|\right)^{(2-p)p/2}\right\|_{2/(2-p),\eta,K\times K} \\ &= C\left(\int_{K\times K}\left(|D_{s,A}u_k|^{p-2}D_{s,A}u_k - |D_{s,A}u|^{p-2}D_{s,A}u\right)\cdot\left(D_{s,A}u_k - D_{s,A}u\right) d\eta\right)^{p/2} = o(1). \end{aligned}$$

This ends the proof of the lemma. ■

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. Let $\varphi \in \mathcal{D}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$. By Lemma 5.3, there holds

$$M([u_k]^A_{s,p})\int_{\mathbb{R}^{2N}} |D_{s,A}u_k|^{p-2}D_{s,A}u_k\overline{D_{s,A}\varphi}\, d\eta + \int_{\mathbb{R}^N} |u_k|^{p-2}u_k\overline{\varphi}V(x)\, dx - \int_{\mathbb{R}^N} |u_k|^{p_s^*-2}u_k\overline{\varphi}\, dx = o(1). \tag{6.29}$$

Firstly, we have that

$$|D_{s,A}u_k|^{p-2}D_{s,A}u_k \text{ is bounded in } L^{p'}_\eta(\mathbb{R}^{2N})$$

and $|D_{s,A}u_k|^{p-2}D_{s,A}u_k \to |D_{s,A}u|^{p-2}D_{s,A}u$ a.e. in $\mathbb{R}^{2N}$ by the previous lemma. Hence,

$$|D_{s,A}u_k|^{p-2}D_{s,A}u_k \rightharpoonup |D_{s,A}u|^{p-2}D_{s,A}u \text{ in } L^{p'}_{\eta}(\mathbb{R}^{2N}).$$

In particular,

$$[u_k]^A_{s,p} \to [u]^A_{s,p}.$$

Therefore, the first integral in (6.29) converges to

$$\mathcal{M}([u]^A_{s,p}) \int_{\mathbb{R}^{2N}} |D_{s,A}u|^{p-2}D_{s,A}u\overline{D_{s,A}\varphi}\, d\eta.$$

The other terms are treated similarly: $|u_k|^{p-2}u_k$ and $|u_k|^{p^*_s-2}u_k$ are bounded in $L^p_V(\mathbb{R}^N)$ and $L^{p^*_s}(\mathbb{R}^N)$, respectively, and converge a.e. to $|u|^{p-2}u$ and $|u|^{p^*_s-2}u$ by [21, Theorem 3.5] and Lemma 6.4, respectively.

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