THREE SOLUTIONS FOR A NONLOCAL PROBLEM WITH CRITICAL GROWTH.

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Abstract. The main goal of this work is to prove the existence of three different solutions (one positive, one negative and one with nonconstant sign) for the equation $(-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda f(x,u)$ in a bounded domain with Dirichlet condition, where $(-\Delta_p)^s$ is the well known p-fractional Laplacian and $p_s^* = \frac{np}{n-sp}$ is the critical Sobolev exponent for the non local case. The proof follows the ideas of [28] and is based in the extension of the Concentration Compactness Principle for the p-fractional Laplacian [20] and Ekeland's variational Principle [7].

1. Introduction

Let us consider the following non local equation with Dirchlet boundary conditions

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

where $s \in (0,1)$, Ω is a smooth and bounded domain in $\mathbb{R}^n$ and $(-\Delta_p)^s u$, called the p-fractional Laplacian, is defined up to a normalization constant by

$$(-\Delta_p)^s u := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\, dy.$$

When $p = 2$ this is the well known fractional Laplacian. Problems involving non local operators have many applications, just to cite a few, we refer to [6, 8, 13] for some physical models, [1, 16, 23] for some applications in finances, [3] for applications in fluid dynamics, [15, 19, 22] for application in ecology and [14] for some applications in image processing.

The functional framework for this operator are the fractional order Sobolev spaces, see [30] and [5]. The fractional order Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) \colon [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ is the famous seminorm of Gagliardo is defined by

$$[u]_{s,p} := \left(\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^p}{|x-y|^{n+ps}}\, dx\, dy \right)^{\frac{1}{p}},$$

and $W_0^{s,p}(\Omega)$ is defined by $W_0^{s,p}(\Omega) := \{u \in L^p(\mathbb{R}^n) : [u]_{s,p} < \infty, u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$. It is well-known that when $sp < n$ the following Sobolev inequality holds

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-sp}}\, dx \right)^{\frac{n-sp}{n}} \leq C \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dxdy$$

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for $u \in C_c^\infty(\mathbb{R}^n)$, where $p_s^* = \frac{np}{n-sp}$ is called the critical Sobolev exponent. So, the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p_s^*$ is continuous. Moreover, is compact for $1 \leq q < p_s^*$. Critical equations with the fractional Laplacian in bounded domains have been considered in [2, 24, 25, 26, 27]. Multiplicity of solutions for nonlocal equation with critical growth was studied in [11, 21]. The main goal of this paper is to show the existence of three different solutions of the problem (1.1). Moreover these solutions are one positive, one negative and one with non constant sign. We impose adequate conditions on the source f and on the parameter λ but we do not impose any parity conditions on the source f. This result extends an old paper of Struwe [29]. Similar results for some local operators can be found in [4, 28, 9, 17]. The method in the proof used in [29] consists on restricting the functional associated to (1.1) to three different manifolds constructed by imposing a sign restriction and normalizing condition. Then using Ekeland variational principle (see [7]) and a generalization to the fractional setting obtained by Mosconi et al. for any $1 < p < \frac{n}{s}$ (see [20]) of the well known Concentration Compactness Principle of P.L.Lions (see [18]), we can prove the existence of a critical point of each restricted functional, that are critical points of the unrestricted one.

Throughout this work, by weak solution of (1.1) we understand critical points of the associated energy functional acting on the Sobolev space $W_0^{s,p}(\Omega)$:

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^p}{|x-y|^{n+ps}}\,dy\,dx - \int_\Omega \frac{1}{p_s^*}|u(x)|^{p_s^*} + \lambda F(x,u(x))\,dx, \tag{1.2}$$

where $F(x,u) = \int_0^u f(x,z)dz$.

2. Assumptions and statement of the results

The precise assumptions on the source terms f are as follows:

- (H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \Omega$.
- (H2) There exist constants $c_1 \in (0, \frac{1}{p_s^*-1})$, $c_2 \in (p, p_s^*)$, $0 < c_3 < c_4$ such that for any $u \in L^q(\Omega)$ and $p < q < p_s^*$,

$$c_3\|u\|^q_{L^q(\Omega)} \leq c_2\int_\Omega F(x,u)\,dx \leq \int_\Omega f(x,u)u\,dx \leq c_1\int_\Omega f_u(x,u)u^2\,dx \leq c_4\|u\|^q_{L^q(\Omega)}.$$

Remark 2.1. The following example fulfill all of our hypotheses, $f(x,u) = |u|^{q-2}u + |u_+|^{r-2}u_+$ if $r \leq q$.

So the main result of the paper reads:

Theorem 2.2. *Under the assumptions* $(H1)-(H2)$, *there exist* $\lambda^* > 0$ *depending only on* n, p, q *and the constant* c_3 *in* $(H2)$, *such that for every* $\lambda > \lambda^*$, *there exist three different, nontrivial, (weak) solutions of problem* (1.1). *Moreover these solutions are, one positive, one negative and the other one has non-constant sing.*

3. Proof of Theorem 2.2

We will construct three disjoint sets K_i not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of C^1-manifolds $M_i \subset W_0^{s,p}(\Omega)$ that will be constructed by imposing a sing restriction and a normalizing condition.

In fact,

Definition 3.1. For each $i = 1, 2, 3$, let $M_i \subset W_0^{s,p}(\Omega)$ be defined as

$$M_1 = \left\{ u \in W_0^{s,p}(\Omega) : \int_\Omega u_+ > 0 \text{ and } [u_+]_{s,p}^p - \int_\Omega |u_+|^{p_s^*}\, dx = \int_\Omega \lambda f(x,u) u_+\, dx \right\},$$

$$M_2 = \left\{ u \in W_0^{s,p}(\Omega) : \int_\Omega u_- > 0 \text{ and } [u_-]_{s,p}^p - \int_\Omega |u_-|^{p_s^*}\, dx = \int_\Omega \lambda f(x,u) u_-\, dx \right\},$$

$$M_3 = M_1 \cap M_2,$$

where $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$.

Definition 3.2. For each $i = 1, 2, 3$, let $K_i \subset W_0^{s,p}(\Omega)$ be defined as

$$K_1 = \{u \in M_1 : u \geq 0\}, \ \ K_2 = \{u \in M_2 : u \leq 0\}, \ \ K_3 = M_3.$$

First, we need the following lemma to show that these sets are nonempty and, moreover, give some properties that will be useful in the proof of our main result.

Lemma 3.3. *For every $w_0 \in W_0^{s,p}(\Omega)$, $w_0 > 0$ ($w_0 < 0$), there exists $t_\lambda > 0$ such that $t_\lambda w_0 \in M_1$ ($\in M_2$). Moreover, $\lim_{\lambda\to\infty} t_\lambda = 0$.*

As a consequence, given $w_0, w_1 \in W_0^{s,p}(\Omega)$, $w_0 > 0$, $w_1 < 0$ with disjoint supports, there exist $\bar{t}_\lambda, \underline{t}_\lambda$ such that $\bar{t}_\lambda w_0 + \underline{t}_\lambda w_1 \in M_3$. Moreover $\bar{t}_\lambda, \underline{t}_\lambda \to 0$ as $\lambda \to \infty$.

Proof. We prove Lemma 3.3 for M_1, the other cases are analogous.

For $w \in W_0^{s,p}(\Omega)$, $w \geq 0$, we consider the functional

$$\varphi_1(w) = [w]_{s,p}^p - \int_\Omega |w|^{p_s^*} + \lambda f(x,w) w\, dx.$$

Given w_0, in order to prove the lemma, we must show that $\varphi_1(t_\lambda w_0) = 0$ for some t_λ. Using the hypothesis (H2), we have that:

$$\varphi_1(tw_0) \geq At^p - Bt^{p_s^*} - \lambda c_4 E t^q$$

and

$$\varphi_1(tw_0) \leq At^p - Bt^{p_s^*} - \lambda c_3 E t^q,$$

where the coefficients A,B and E are given by:

$$A = [w_0]_{s,p}^p\,, \ B = \int_\Omega |w_0|^{p_s^*}\, dx, \ E = \int_\Omega |w_0|^q\, dx.$$

Since $p < q < p_s^*$ it follows that $\varphi_1(tw_0)$ is positive for a t small enough, and negative for t big enough. Hence, by Bolzano's Theorem, there exists some $t = t_\lambda$ such that $\varphi_1(t_\lambda w_0) = 0$.

In order to give an upper bound for t_λ, it is enough to find some t_1, such that $\varphi_1(t_1 w_0) < 0$. We observe that:

$$\varphi_1(tw_0) < At^p - \lambda c_3 E t^q,$$

so it is enough to choose t_1 such that $At_1^p - \lambda c_3 E t_1^q = 0$, i.e.,

$$t_1 = \left(\frac{A}{c_3 \lambda E} \right)^{\frac{1}{(q-p)}},$$

therefore, again by Bolzano's Theorem, we can choose $t_\lambda \in [0, t_1]$, which implies that $t_\lambda \to 0$ when $\lambda \to +\infty$, as we wanted to prove. □

For the proof of Theorem 2.2, we need also the following lemmas.

Lemma 3.4. *There exist constants $\alpha_j > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,*

$$\alpha_1 [u]^p_{s,p} \leq \alpha_2 \left(\int_\Omega |u|^{p^*_s} + \lambda f(x,u) u \, dx \right) \leq \alpha_3 \Phi(u) \leq \alpha_4 [u]^p_{s,p}.$$

Proof. As $u \in K_i$, we have that

$$[u]^p_{s,p} = \int_\Omega |u|^{p^*_s} + \lambda f(x,u) u \, dx,$$

choosing $\alpha_1 = \alpha_2$ we have the first inequality.

For the last inequality by (H2)

$$\int_\Omega F(x,u) \, dx \leq \frac{1}{c_2} \int_\Omega f(x,u) u \, dx.$$

Furthermore,

$$\left| \lambda \int_\Omega F(x,u) \, dx \right| = \lambda \int_\Omega F(x,u) \, dx \leq \frac{1}{c_2} \int_\Omega \lambda f(x,u) u \, dx = \frac{1}{c_2} \left([u]^p_{s,p} - \int_\Omega |u|^{p^*_s} \, dx \right),$$

so

$$-\lambda \int_\Omega F(x,u) \, dx \leq \frac{1}{c_2} \left([u]^p_{s,p} - \int_\Omega |u|^{p^*_s} \, dx \right). \tag{3.1}$$

By 3.1, we have:

$$\begin{aligned} \Phi(u) =& \frac{1}{p} [u]^p_{s,p} - \int_\Omega \frac{1}{p^*_s} |u(x)|^{p^*_s} + \lambda F(x,u) \, dx \\ \leq& \frac{1}{p} [u]^p_{s,p} - \int_\Omega \frac{1}{p^*_s} |u(x)|^{p^*_s} \, dx + \frac{1}{c_2} \left([u]^p_{s,p} - \int_\Omega |u(x)|^{p^*_s} \, dx \right) \\ \leq& \frac{1}{p} [u]^p_{s,p} + \frac{1}{c_2} [u]^p_{s,p} \\ \leq& \left(\frac{1}{p} + \frac{1}{c_2} \right) [u]^p_{s,p}. \end{aligned}$$

This proves the third inequality, with $\alpha_4 = \left(\frac{1}{p} + \frac{1}{c_2} \right) \alpha_3$.

To prove the middle inequality we proceed as follows:

$$\begin{aligned} \Phi(u) =& \frac{1}{p} [u]^p_{s,p} - \int_\Omega \frac{1}{p^*_s} |u(x)|^{p^*_s} + \lambda F(x,u) \, dx \\ \geq& \frac{1}{p} [u]^p_{s,p} - \int_\Omega \frac{1}{p^*_s} |u(x)|^{p^*_s} \, dx - \frac{1}{c_2} \int_\Omega \lambda f(x,u) u \, dx. \end{aligned}$$

So

$$\begin{aligned}c_2\Phi(u) \geq& c_2\frac{1}{p}[u]^p_{s,p} - c_2\int_\Omega \frac{1}{p^*_s}|u(x)|^{p^*_s}\,dx - \int_\Omega \lambda f(x,u)u\,dx\\ =& c_2\frac{1}{p}\left(\int_\Omega |u(x)|^{p^*_s}\,dx + \int_\Omega \lambda f(x,u)u\,dx\right) - c_2\int_\Omega \frac{1}{p^*_s}|u(x)|^{p^*_s}\,dx - \int_\Omega \lambda f(x,u)u\,dx\\ =& c_2\frac{1}{p}\int_\Omega |u(x)|^{p^*_s}dx + c_2\frac{1}{p}\int_\Omega \lambda f(x,u)u\,dx - c_2\int_\Omega \frac{1}{p^*_s}|u(x)|^{p^*_s}\,dx - \int_\Omega \lambda f(x,u)u\,dx\\ =& c_2\left(\frac{1}{p}-\frac{1}{p^*_s}\right)\int_\Omega |u(x)|^{p^*_s}\,dx + \left(c_2\frac{1}{p}-1\right)\int_\Omega \lambda f(x,u)u\,dx.\end{aligned}$$

Since $\gamma_1 = c_2\left(\frac{1}{p}-\frac{1}{p^*_s}\right)$ and $\gamma_2 = \left(c_2\frac{1}{p}-1\right)$ are positive, we take $\alpha_2 = \min\{\gamma_1,\gamma_2\}$, $\alpha_3 = c_2$ and we have

$$\alpha_3\Phi(u) \geq \alpha_2\left(\int_\Omega |u|^{p^*_s} + \lambda f(x,u)u\,dx\right).$$

This finishes the proof. □

Lemma 3.5. *There exists a constant D such that $[u_+]^p_s \geq D$, for all $u \in K_1$, $[u_-]^p_{s,p} \geq D$ for all $u \in K_2$, and $[u_-]^p_{s,p}$, $[u_+]^p_s \geq D$ for all $u \in K_3$.*

Proof. By definition of K_i we have

$$[u_\pm]^p_{s,p} = \|u_\pm\|^{p^*_s}_{p^*_s} + \int_\Omega \lambda f(x,u)u_\pm\,dx.$$

Using (H2) we have

$$\int_\Omega \lambda f(x,u)u_\pm\,dx \leq c_4\|u_\pm\|^q_q, \text{ for } p^*_s \geq q > p.$$

Then

$$[u_\pm]^p_{s,p} \leq \|u_\pm\|^{p^*_s}_{p^*_s} + c_4\|u_\pm\|^q_q \leq \tilde{C}\left([u_\pm]^{p^*_s}_s + [u_\pm]^q_s\right).$$

In the second inequality we use Poincaré inequality. In summary $[u_+]^p_{s,p} \leq \hat{C}[u_\pm]^r_{s,p}$. Where $r = q$ if $[u_\pm]_{s,p} < 1$ or $r = p^*_s$ if $[u_\pm]_{s,p} \geq 1$. Since $r > p$ we have what we need. □

The following lemma describes the properties of the manifolds M_i.

Lemma 3.6. *M_i is a sub-manifold of $W^{s,p}_0(\Omega)$ of codimension 1, if $i = 1, 2$ and 2 if $i = 3$ respectively, the sets K_i are complete, and for every $u \in M_i$ we have $T_uW^{s,p}_0(\Omega) = T_uM_i \oplus span\{u_+, u_-\}$ where T_uM is the tangent space at u of the Banach manifold M. Finally, the projection to the first coordinate is uniformly continuous on M_i.*

Proof. We consider

$$\overline{M_1} = \left\{u \in W^{s,p}_0(\Omega) : \int_\Omega u_+ > 0\right\},$$

$$\overline{M_2} = \left\{u \in W^{s,p}_0(\Omega) : \int_\Omega u_- > 0\right\},$$

$$\overline{M_3} = \overline{M_1} \cap \overline{M_2}.$$

Observe that $M_i \subset \overline{M_i}$ and since the sets $\overline{M_i}$ are open so it's sufficient to prove that M_i is a regular sub-manifold of $W^{s,p}_0(\Omega)$.

We are going to build a function C^1, $\varphi : \overline{M_i} \to \mathbb{R}^d$ with $d = 1$ if $i = 1, 2$ or $d = 2$ if $i = 3$, such that M_i is the inverse of a regular value of φ_i.

We define

$$\varphi_1(u) = [u_+]^p_{s,p} - \int_\Omega |u_+|^{p^*_s} + \lambda f(x,u)u_+ \, dx \quad \text{for } u \in M_1,$$

$$\varphi_2(u) = [u_-]^p_{s,p} - \int_\Omega |u_-|^{p^*_s} + \lambda f(x,u)u_- \, dx \quad \text{for } u \in M_2,$$

and

$$\varphi_3(u) = (\varphi_1(u), \varphi_2(u)) \quad \text{for } u \in M_3.$$

We have that $M_i = \varphi_i^{-1}(0)$ so we have to prove that 0 is a regular value of φ_i.

Let us calculate $\langle \nabla\varphi_1(u), u_+\rangle$ for $u \in M_1$,

$$\frac{d}{d\varepsilon}\varphi_1(u+\varepsilon u_+) = \frac{d}{d\varepsilon}\left([(u+\varepsilon u_+)_+]^p_{s,p} - \int_\Omega |(u+\varepsilon u_+)_+|^{p^*_s} + \lambda f(x, u+\varepsilon u_+)(u+\varepsilon u_+)_+ \, dx\right).$$

Since $(u+\varepsilon u_+)_+ = u_+ + \varepsilon u_+$ we have that $\frac{d}{d\varepsilon}\varphi_1(u+\varepsilon u_+)$ is equal to

$$(1+\varepsilon)^{p-1} p[u_+]^p_{s,p} - \int_\Omega p^*_s(1+\varepsilon)^{p^*_s-1}|u_+|^{p^*_s} + \lambda f(x, u+\varepsilon u_+)u_+ + \lambda f_u(x, u+\varepsilon u_+)(1+\varepsilon)u_+^2 \, dx,$$

then since $u \in M_1$,

$$\begin{aligned}
\frac{d}{d\varepsilon}\varphi_1(u+\varepsilon u_+)\Big|_{\varepsilon=0} &= \left(p\,[u_+]^p_{s,p} - \int_\Omega p^*_s|u_+|^{p^*_s} + \lambda f(x,u)u_+ + \lambda f_u(x,u)u_+^2 \, dx\right)\\
&\le p^*_s\left([u_+]^p_{s,p} - \int_\Omega |u_+|^{p^*_s}\, dx\right) - \int_\Omega \lambda f(x,u)u_+ + \lambda f_u(x,u)u_+^2 \, dx\\
&= p^*_s\left(\int_\Omega \lambda f(x,u)u_+ dx\right) - \int_\Omega \lambda f(x,u)u_+ + \lambda f_u(x,u)u_+^2 \, dx\\
&= (p^*_s - 1)\left(\int_\Omega \lambda f(x,u)u_+ \, dx\right) - \int_\Omega \lambda f_u(x,u)u_+^2 \, dx.
\end{aligned}$$

By (H_2) we know that there exists $c_1 \in \left(0, \frac{1}{p^*_s-1}\right)$ such that

$$\int_\Omega \lambda f(x,u)u_+ \, dx \le c_1 \int_\Omega \lambda f_u(x,u)u_+^2 \, dx. \tag{3.2}$$

Then

$$(p^*_s - 1)\int_\Omega \lambda f(x,u)u_+ \, dx - \int_\Omega \lambda f_u(x,u)u_+^2 \, dx < 0.$$

In summary, we have that $\langle \nabla\varphi_1(u), u_+\rangle < 0$, then $\nabla\varphi_1(u) \neq 0$. This means that M_1 is a regular submanifold of $W^{s,p}_0(\Omega)$.

The proof for M_2, is analogous.

Let's observe that if we prove that $\langle \nabla\varphi_2(u), u_+\rangle = \langle \nabla\varphi_1(u), u_-\rangle = 0$ for $u \in M_3$ then for what we had made before, we know that $\langle \nabla\varphi_1(u), u\rangle < 0$ and $\langle \nabla\varphi_2(u), u\rangle < 0$. For this we can affirm that $\nabla\varphi_3(u) \neq 0$ for $u \in M_3$.

Then we will prove that $\langle \nabla\varphi_1(u), u_-\rangle = 0$. In fact,

$$\begin{aligned}\frac{d}{d\varepsilon}\varphi_1(u+\varepsilon u_-) =& \frac{d}{d\varepsilon}\left([(u+\varepsilon u_-)_+]^p_{s,p} - \int_\Omega |(u+\varepsilon u_-)_+|^{p^*_s} + \lambda f(x,u+\varepsilon u_-)(u+\varepsilon u_-)_+\, dx \right)\\ =& \frac{d}{d\varepsilon}\left([u_+]^p_{s,p} - \int_\Omega |u_+|^{p^*_s} + \lambda f(x,u+\varepsilon u_-)u_+\, dx \right)\\ =& -\int_\Omega \lambda f_u(x,u+\varepsilon u_-)u_+u_-\, dx = 0.\end{aligned}$$

Then

$$\left.\frac{d}{d\varepsilon}\varphi_1(u+\varepsilon u_-)\right|_{\varepsilon=0} = 0.$$

In an analogous way we have $\langle \nabla\varphi_2(u), u_+\rangle = 0$. Therefore, M_3 is a regular submanifold.

The completeness of K_i is easy and is left to the reader.

Finally, it remains to see that

$$T_u W^{s,p}_0(\Omega) = T_u M_1 \oplus \operatorname{span}\{u_+\},$$

where $M_1 = \{u : \varphi_1(u) = 0\}$ and $T_u M_1 = \{v : \langle \nabla\varphi_1(u), v\rangle = 0\}$. Now let $v \in T_u W^{s,p}_0(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where $v_2 = \alpha u_+$ and $v_1 = v - v_2$. Let us take α as

$$\alpha = \frac{\langle \nabla\varphi_1(u), v\rangle}{\langle \nabla\varphi_1(u), u_+\rangle}.$$

With this choice, we have that $v_1 \in T_u M_1$. Now

$$\langle \nabla\varphi_1(u), v_1\rangle = 0.$$

The very same argument is used to show that $T_u W^{s,p}_0(\Omega) = T_u M_2 \oplus \operatorname{span}\{u_-\}$ and $T_u W^{s,p}_0(\Omega) = T_u M_i \oplus \operatorname{span}\{u_+, u_-\}$.

From these formulas and the estimates given in the first part of the proof, the uniform continuity of the projections onto $T_u M_i$ follows. $\square$

Now, we say that $\{u_j\} \subset W^{s,p}_0(\Omega)$ is a Palais-Smale sequence of c level if

(i) $\Phi(u_j) \to c$,
(ii) $\nabla\Phi(u_j) \to 0$ in $W^{-s,p}(\Omega)$.

We say that Φ satisfies Palais-Smale condition of level c if for every $\{u_j\}$ Palais-Smale sequence of level c there exists a subsequence that converges strongly in $W^{s,p}_0(\Omega)$.

Now, in order to use Ekeland's variational principle, we need to check the Palais-Smale condition for the functional Φ restricted to the manifold M_i. To this end, we need the following lemma which proves the Palais-Smale condition for the unrestricted functional below certain energy level.

Lemma 3.7. *The unrestricted functional Φ verifies the Palais-Smale condition for energy level c for every $c < \frac{s}{n}S^{\frac{n}{sp}}$, where S is the best Sobolev constant for the fractional Laplacian $S := \inf_{\phi\in C^\infty_c(\Omega)} \frac{[\phi]^p_{s,p}}{\|\phi\|^p_{p^*_s}}$.*

The proof of Lemma 3.7 is omitted as it uses standard ideas and is based in the Concentration Compactness Principle for nonlocal operators (see[20]). For the local case it can be found in [12, 28]. For the non local case it follows similarly, see [10] for the details.

Now, we can prove the Palais-Smale condition for the restricted functional.

Lemma 3.8. *The functional $\Phi|_{K_i}$ satisfies the Palais-Smale condition for energy level c for every $c < \frac{s}{n} S^{\frac{n}{sp}}$.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi|_{K_i} \to 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i.

Let $v_j \in T_{u_j} W_0^{s,p}(\Omega)$ be a unit tangential vector such that

$$\langle \nabla\Phi(u_j), v_j\rangle = \|\nabla\Phi(u_j)\|_{W_0^{-s,p}(\Omega)}.$$

Now, by lemma 3.6, $v_j = w_j + z_j$ with $w_j \in T_{u_j} M_i$ and $z_j \in span\{(u_j)_+, (u_j)_-\}$.

Since $\Phi(u_j)$ is uniformly bounded, by Lemma 3.4, u_j is uniformly bounded in $W_0^{s,p}(\Omega)$ and hence w_j is uniformly bounded in $W_0^{s,p}(\Omega)$. Therefore

$$\|\nabla\Phi(u_j)\|_{W_0^{-s,p}(\Omega)} = \langle \nabla\Phi(u_j), v_j\rangle = \langle \nabla\Phi|_{K_i}(u_j), v_j\rangle \to 0.$$

As v_j is uniformly bounded and $\nabla\Phi|_{K_i}(u_j) \to 0$ strongly, the inequality converges strongly to 0. Now the result follows by Lemma 3.7.

□

We now immediately obtain the following lemma.

Lemma 3.9. *There exists $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Moreover u is also a critical point of the unrestricted functional Φ and hence a weak solution to* (1.1).

With all this preparatives, this is the proof of our main result.

Proof of Theorem 2.2. To prove the Theorem 2.2, we need to check that the functional $\Phi|_{K_i}$ verifies the hypotheses of the Ekeland's Variational Principle.

The fact that Φ is bounded below over K_i is a direct consequence of the construction of the manifold K_i.

Then by Ekeland's Variational Principle, there exists $v_k \in K_i$, such that

$$\Phi(v_k) \to c_i \text{ and } (\nabla\Phi|_{K_i})(v_k) \to 0.$$

We have to check that if we choose λ large, we have that $c_i < \frac{s}{n} S^{\frac{n}{sp}}$. This follows easily from Lemma 3.3. For instance, for c_1 we have that choosing $w_0 \geq 0$,

$$c_1 \leq \Phi(t_\lambda w_0) \leq \frac{1}{p} t_\lambda^p [w_0]_{s,p}^p.$$

Moreover, it follows from the estimate of t_λ in Lemma 3.3 , that $c_1 \to 0$ as $\lambda \to 0$. Then $c_i < \frac{s}{n} S^{\frac{n}{sp}}$ for $\lambda > \lambda^*(p, q, n, c_3)$. The other cases are analogous.

From Lemma 3.7, it follows that v_k has a convergent subsequence, that we still call v_k. Therefore Φ has a critical point in K_i, $i = 1, 2, 3$ and, by construction, one of them is positive, other is negative and the last one changes sign. □

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