EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR THE FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - f(u) = 0 \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $p \in \left(1, \frac{N+2s}{N-2s}\right)$, $N > 2s$, $f(u) = |u|^{p-1}u$, $V \in L^\infty(\mathbb{R}^N)$ is such that $\inf_{\mathbb{R}^N} V > 0$ and $\varepsilon > 0$ is small. We study the existence and nonexistence of solutions concentrating at a local minimum point of V as $\varepsilon \to 0$ without using any symmetry assumption on V. First, we prove that certain type of positive solutions exhibiting peaks do not exist. Then, we study the existence of sign-changing solutions under a suitable configuration of positive and negative peaks. To guarantee the existence, we cannot neglect the interaction between peaks. In particular, by using a minimization argument, we found solutions exhibiting peaks at the vertices of a 2ℓ-regular polygon, such that two adjacent peaks have alternate sign.

1. INTRODUCTION

This paper concerns the existence and nonexistence of solutions of the equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $p \in \left(1, \frac{N+2s}{N-2s}\right)$, $N > 2s$, $f(t) = |t|^{p-1}t$, $V \in L^\infty(\mathbb{R}^N)$ is such that $\inf_{\mathbb{R}^N} V > 0$ and $\varepsilon > 0$ is small.

It is well known that solutions of (1.1) give rise to standing wave solutions of the fractional nonlinear Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \hbar^{2s}(-\Delta)^s \psi + U(x)\psi - |\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \times (0, +\infty), \tag{1.2}$$

in the semiclassical limit regime $0 < \varepsilon := \hbar \ll 1$, where $U(x)$ is a bounded potential and $\hbar$ denotes the usual Planck constant. Indeed, standing wave solutions of (1.2) have the form $\psi(x,t) = u(x)e^{\frac{\mathrm{i}Et}{\varepsilon}}$, where u represents a real-valued function. Then, by letting $V(x) = U(x) + E$, one can check directly that such a function u actually satisfies (1.1). Equation (1.2) was introduced by Laskin [16, 17] as a generalization of the classical nonlinear Schrödinger equation where the Brownian trajectories that lead to standard quantum and statistical mechanics are replaced by the Lévy paths, leading to fractional quantum and fractional statistical mechanics.

Throughout this paper, $\mathscr{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decaying smooth functions, and the fractional Laplacian $(-\Delta)^s$, with $s \in (0,1)$, of a function

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$\vartheta \in \mathscr{S}(\mathbb{R}^N)$ is defined by

$$(-\Delta)^s\vartheta(x) := \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(\vartheta)(\xi)\right)(x) \quad \text{for all } x \in \mathbb{R}^N,$$

where $\mathscr{F}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ its inverse. Additionally, we consider the space $H^s(\mathbb{R}^N)$, which is a natural space for solutions of (1.1) and can be defined in an alternative way via a Fourier transform by the space

$$\widehat{H}^s(\mathbb{R}^N) := \left\{\vartheta \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N}(1+|\xi|^{2s})|\mathscr{F}(\vartheta)(\xi)|^2 d\xi < \infty\right\}.$$

Continuing the route initiated by Amick and Toland [6] for the study of the Benjamin-Ono equation, *i.e.*, the equation (1.1) with $s = \frac{1}{2}$ and $N = 1$, Frank and Lenzmann [13] studied the uniqueness of positive solutions to (1.1) in the one-dimensional case for $s \in (0,1)$, obtaining nondegeneracy and symmetry of solutions. These results were extended to $N \geq 1$ in [14], getting also uniqueness for radial solutions and nondegeneracy of ground state solutions.

In recent years, many results regarding concentration phenomena for equation (1.1) and its generalizations, under the assumption that $\inf_{\mathbb{R}^N} V > 0$ have arisen; see, for instance, [2, 3, 4, 5, 9, 8, 11, 24, 23, 1, 19, 18, 25]. In particular, in [19, 23, 25], multipeak solutions were studied by overlapping single peaks that are sufficiently far away from one another so that one peak has no effect on the other peaks in the areas where decay occurs, avoiding interactions between peaks.

Here, we are interested in the case where the interactions between peaks are essential to building clustered solutions to (1.1) with peaks approaching at the same point. For the case $s = 1$, this phenomenon was studied for the first time by Kang and Wei [15]. For the case $s \in (0,1)$, we know of two works motivating our study. Positive solutions exhibiting ℓ interacting peaks concentrating at a local maximum of the potential V were found by Dávila, Del Pino and Wei [9] through the Lyapunov-Schmidt variational reduction. Also via a reduction scheme, Long and Lv [18] constructed sign-changing solutions concentrating at a local minimum of the potential V under certain symmetry assumptions on V. To the best of our knowledge, those results seem to be the only ones available concerning sign-changing solutions to (1.1) in the literature. In both works, the interaction between the peaks cannot be neglected, since it plays a key role in guaranteeing the existence of the solutions. Our main goals here are to study situations other than those in [9, 18]. Specifically, without any symmetry assumption regarding the potential V, we first prove the nonexistence of positive solutions of such type concentrating at a local minimum of the potential V and then, we study the issue of the existence of sign-changing solutions exhibiting interacting peaks under appropriate configurations of concentration points.

To put our results into perspective, in the remainder of this paper, we consider the function $w \in H^s(\mathbb{R}^N)$ being the unique positive radial ground state solution of

$$\begin{cases} (-\Delta)^s w + w - f(w) = 0 & \text{in } \mathbb{R}^N, \\ w(0) = \max_{\mathbb{R}^N} w, \quad w(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

in which $f(t) = |t|^{p-1}t$ for all $t \in \mathbb{R}$, $s \in (0,1)$, $p \in \left(1, \frac{N+2s}{N-2s}\right)$ and $N > 2s$, see [14]. It is known that such function w satisfies

$$w(z) = \frac{\gamma_0(1+o(1))}{|z|^{N+2s}}, \quad \text{where } o(1) \to 0 \text{ as } |z| \to \infty \text{ for some } \gamma_0 > 0. \tag{1.3}$$

We point out that this asymptotic behavior differs from the one known for the solutions to the problem in the case $s = 1$, which is an exponential type decay at infinity. Despite this fact, the decay (1.3) will be sufficient and crucial for our purposes although we will have to do accurate estimates to cover the case

$0<s<1$. Observe that for a fixed $\lambda>0$, the function $w_\lambda(y) := \lambda^{\frac{1}{p-1}} w(\lambda^{\frac{1}{2s}} y)$, $y\in\mathbb{R}^N$, belongs to $H^{2s+1}(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)$ and verifies

$$(-\Delta)^s\vartheta + \lambda\vartheta - f(\vartheta) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

which, after a suitable change in variables, can be seen as a limit equation coming from (1.1). Moreover, the function w_λ, among all nontrivial solutions of (1.4) in $H^{2s}(\mathbb{R}^N)$, yields the lowest possible value for the energy functional I_λ defined by

$$I_\lambda(\widetilde{v}) := \frac{1}{2}\int_{\mathbb{R}^N} (v(-\Delta)^s v + \lambda v^2)\,dx - \int_{\mathbb{R}^N} F(v)\,dx, \tag{1.5}$$

where $F(t)=\int_0^t f(s)\,ds = \frac{1}{p+1}|t|^{p+1}$ and $\widetilde{v}$ is the s-harmonic extension of $v\in H^{2s}(\mathbb{R}^N)$. Then, it is reasonable to search for solutions u_ε of (1.1) that resemble

$$u_\varepsilon(x) \sim \sum_{j=1}^{\ell'} \lambda_j^{\frac{1}{p-1}} w\Big(\lambda_j^{\frac{1}{2s}}\Big(\frac{x-Q_j}{\varepsilon}\Big)\Big) \pm \sum_{j=\ell'+1}^{\ell} \lambda_j^{\frac{1}{p-1}} w\Big(\lambda_j^{\frac{1}{2s}}\Big(\frac{x-Q_j}{\varepsilon}\Big)\Big), \quad x\in\mathbb{R}^N,$$

for a suitable choice of points $Q_1,\dots,Q_\ell$ close to the same critical point Q_0 of V, and certain positive values $\lambda_1,\dots,\lambda_\ell$ close to $V(Q_0)$. We are interested in the case in which Q_0 is a local minimum point of V. To continue, we introduce the function

$$w_\lambda^Q(x) := \lambda^{\frac{1}{p-1}} w\Big(\lambda^{\frac{1}{2s}}\Big(\frac{x-Q}{\varepsilon}\Big)\Big), \quad x\in\mathbb{R}^N,$$

where $Q\in\mathbb{R}^N$ and $\lambda>0$; and proper hypotheses regarding the potential V. Specifically, we assume that the following statements hold:

- (V_0) $V\in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $\inf_{\mathbb{R}^N} V>0$.
- (V_1) There exists an open bounded smooth domain $\Omega\subset\mathbb{R}^N$ such that $V\in C^1(\Omega)$, and there exists unique $Q_0\in\Omega$ such that $V(Q_0)=\inf_\Omega V<\inf_{\partial\Omega} V$.
- (V_2) There exists an open set ω compactly contained in Ω such that $Q_0\in\operatorname{int}(\omega)$, $V\in C^{1,\theta}(\omega)$ for some $\theta\in(0,1)$, and $V(Q)>V(Q_0)$ for all $Q\in\omega\setminus\{Q_0\}$.

Our first main result concerns the nonexistence of certain positive solutions concentrating near a nondegenerate local minimum point of the potential V. Of course, this result extends to the fractional case the Theorem 1.2 in [15].

Theorem 1.1. *Let $N>1$ and $\ell\in\mathbb{N}$, $\ell>1$. Assume that V satisfies (V_0), (V_1), and (V_2), with $V\in C^2(\omega)$ and $\det(D^2V(Q_0))\neq 0$. Then, there exists $\varepsilon_0>0$ such that for each $0<\varepsilon<\varepsilon_0$ the equation (1.1) cannot have a positive solution u_ε of the form*

$$u_\varepsilon(x) = \sum_{i=1}^{\ell} w_{\lambda_i^\varepsilon}^{Q_i^\varepsilon}(x) + \varphi_\varepsilon(x), \quad x\in\mathbb{R}^N, \tag{1.6}$$

where $\varphi_\varepsilon\in H^{2s}(\mathbb{R}^N)$, with $\varphi_\varepsilon\to 0$ on $H^{2s}(\mathbb{R}^N)$ as $\varepsilon\to 0$, and $\lambda_i^\varepsilon = V(Q_i^\varepsilon)\to V(Q_0)$ as $\varepsilon\to 0$ and $\min_{i\neq j}\left|Q_i^\varepsilon - Q_j^\varepsilon\right| > \varepsilon^{1-\frac{\rho+2}{N+2s}}$ and $Q_i^\varepsilon\to Q_0$ as $\varepsilon\to 0$ for all $i=1,\dots,\ell$, where $0\le\rho<N+2s-2$ is given.

The proof is based on the study of a system of equations on the locations of the peaks, joint an argument that relies on the nondegeneracy of the potential V on Q_0.

Our second main result refers to the existence of sign-changing solutions that exhibit the same numbers of positive and negative peaks.

Theorem 1.2. *Let $N>1$ and $\ell\in\mathbb{N}$. Assume that (V_0), (V_1) and (V_2) hold. Then, for each $\varepsilon>0$ that is sufficiently small, there exists a solution $u_\varepsilon\in H^{2s}(\mathbb{R}^N)$ of (1.1) such that*

$$u_\varepsilon(x) = \sum_{i=1}^{\ell} w_{\lambda_i^\varepsilon}^{Q_i^\varepsilon}(x) - \sum_{i=\ell+1}^{2\ell} w_{\lambda_i^\varepsilon}^{Q_i^\varepsilon}(x) + \varphi_\varepsilon(x), \quad x\in\mathbb{R}^N, \tag{1.7}$$

where $\varphi_\varepsilon \in H^{2s}(\mathbb{R}^N)$*, with* $\varphi_\varepsilon \to 0$ *on* $H^{2s}(\mathbb{R}^N)$ *as* $\varepsilon \to 0$*, and* $\lambda_i^\varepsilon = V(Q_i^\varepsilon)$*, with* $Q_i^\varepsilon \in \omega$ *satisfying* $V(Q_i^\varepsilon) \to V(Q_0)$ *as* $\varepsilon \to 0$*. Moreover,*

$$\min_{i\neq j} \left|Q_i^\varepsilon - Q_j^\varepsilon\right| > \varepsilon^{\frac{N+s}{N+2s}} \quad \text{and} \quad Q_i^\varepsilon \to Q_0 \quad \text{as } \varepsilon \to 0 \quad \text{for all } i = 1, \dots, 2\ell.$$

The proof is based on a minimization argument where the term representing the interaction between peaks has to be precisely controlled, which is achieved by choosing a configuration of points involving the same number of positive and negative peaks. This argument can be compared with that of maximizing in [9, Theorem 3] related the existence of multiple positive peaks concentrating at a local maximum point of V verifying $\min_{i\neq j} \left|Q_i^\varepsilon - Q_j^\varepsilon\right| > \varepsilon^{\frac{4-s}{4}}$. Unlike that work, here we need more precise estimates of the error (compare our Lemma 4.1 with [9, Lemma 6.4]) and to choose the concentration points in a very special form in order to obtain the desired effect of the interaction between peaks. Namely, in our argument we will choose points Q_i^ε as the vertices of a regular 2ℓ-polygon centered at Q_0 so that two adjacent peaks have alternate sign, and then will obtain the desired result. We emphasize that, under our hypothesis, such configuration of concentration points has not been considered in the literature yet, even for the case $s = 1$.

The proof of our theorems relies on the Lyapunov-Schmidt reduction method, which reduces the problem of finding $u \in H^{2s}(\mathbb{R}^N)$ that solves (1.1) to finding a critical point $\mathbf{q} \in \mathbb{R}^{N\ell}$, where $\mathbf{q} = \varepsilon^{-1}(Q_1, \dots, Q_\ell) \in \mathbb{R}^{N\ell}$, for a function denoted by $\mathcal{J}_\varepsilon$. This latter comes from evaluating the energy functional J_ε associated to

$$(-\Delta)^s\vartheta + V(\varepsilon x)\vartheta - f(\vartheta) = 0 \quad \text{in } \mathbb{R}^N \tag{1.8}$$

in a constructed solution that, roughly speaking, has the form $v_\varepsilon(x) = W(x) + \phi_\varepsilon(x)$, $x \in \mathbb{R}^N$, with $W(x) = \sum_{i=1}^{\ell} \tau_i w_{\lambda_i}^{Q_i}(\varepsilon x)$, $\tau_i \in \{-1, 1\}$, and the remainder term ϕ_ε is of lower order than W respect to a weighted norm. It is, $\mathcal{J}_\varepsilon(\mathbf{q}) := J_\varepsilon(\widetilde{v}_\varepsilon)$, where $\widetilde{v}_\varepsilon$ is the s-harmonic extension of v_ε for suitable points $\mathbf{q} \in \mathbb{R}^{N\ell}$. Since finding critical points of $\mathcal{J}_\varepsilon$ becomes equivalent to finding solutions to (1.8), our effort shall be fully devoted to finding critical points of $\mathcal{J}_\varepsilon$. The reduction procedure used here was devised in [10] for a slightly supercritical problem involving the Laplacian operator in a bounded domain; see also [12, 21, 22, 26], among other pioneering works in which this method has been implemented for the case $s = 1$. Additionally, we also consider some ideas introduced in [15, 9] related to the interactions between peaks.

In Section 2, we sketch the reduction procedure. Section 3 is dedicated to proving Theorem 1.1, whereas in Section 4 we give the proof of Theorem 1.2. Finally, we include two appendix to exhibit some technical details related the reduction procedure.

2. The reduction procedure

2.1. **Functional framework and preliminaries.** In this subsection, we offer a brief review of the fractional Sobolev spaces in the context of our problem. Let $\mathscr{S}(\mathbb{R}^N)$ be the Schwartz space of rapidly decaying smooth functions, *i.e.*,

$$\mathscr{S}(\mathbb{R}^N) := \left\{\vartheta \in C^\infty(\mathbb{R}^N) \mid \sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta \vartheta(x)| < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^N\right\}.$$

As already mentioned in the introduction, here, we consider the fractional Laplacian $(-\Delta)^s$, with $s \in (0, 1)$, of a function $\vartheta \in \mathscr{S}(\mathbb{R}^N)$ defined as

$$(-\Delta)^s\vartheta(x) := \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(\vartheta)(\xi)\right)(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{2.1}$$

where $\mathscr{F}$ denotes the Fourier transform, *i.e.*,

$$\mathscr{F}(\vartheta)(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\mathrm{i}\xi\cdot x}\vartheta(x)\, dx \quad \text{for all } \xi \in \mathbb{R}^N,$$

and $\mathscr{F}^{-1}$ its inverse. For later purposes and by taking advantage of the fact that we work on $\mathscr{S}(\mathbb{R}^N)$, we invoke a suitable characterization of the fractional Laplacian operator through the extension operator that was introduced by Caffarelli and Silvestre [7] which is equivalent to (2.1). The importance of this extension is that it will allow us to solve (1.1) using variational methods. We now briefly describe this extension. For $\vartheta \in H^s(\mathbb{R}^N)$, where

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) \mid \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx\, dy < \infty \right\},$$

or equivalently $H^s(\mathbb{R}^N) = \widehat{H}^s(\mathbb{R}^N)$ (see for example [20]), we consider the boundary value problem

$$\begin{cases} \operatorname{div}\big(t^{1-2s}\nabla\widetilde{\vartheta}\big) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \widetilde{\vartheta}(\cdot,0) = \vartheta & \text{on } \mathbb{R}^N, \end{cases}$$

with $\mathbb{R}^{N+1}_+ := \{(x,t) \mid x \in \mathbb{R}^N, t > 0\}$ and $\widetilde{\vartheta}$ is the s-harmonic extension of the function ϑ given by

$$\widetilde{\vartheta}(x,t) := \int_{\mathbb{R}^N} \mathscr{P}_s(x-y,t)\,\vartheta(y)\,dy,$$

where $\mathscr{P}_s$ is the generalized Poisson kernel of order s given by $\mathscr{P}_s(x,t) := \frac{1}{t^N}\mathscr{K}_s\big(\frac{x}{t}\big)$ for all $(x,t) \in \mathbb{R}^{N+1}_+$, with $\mathscr{K}_s(z) := c_{N,s}(1+|z|^2)^{-\frac{N+2s}{2}}$ for all $z \in \mathbb{R}^N$, where the constant $c_{N,s} > 0$ is chosen such that $\int_{\mathbb{R}^N} \mathscr{K}_s(z)\,dz = 1$ holds. Then, $(-\Delta)^s\vartheta$ can be obtained as the Dirichlet-to-Neumann map for this problem, namely,

$$(-\Delta)^s\vartheta(x) = -b_s \lim_{t\to 0^+} t^{1-2s}\,\partial_t\widetilde{\vartheta}(x,t) \quad \text{for all } x \in \mathbb{R}^N, \tag{2.2}$$

where $b_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$. Now, let $m > 0$ and $g \in L^2(\mathbb{R}^N)$. It is known that the equation $(-\Delta)^s\vartheta + m\vartheta = g$ in $\mathbb{R}^N$ has a unique solution $\psi \in H^{2s}(\mathbb{R}^N)$ given by

$$\psi(x) = (\mathcal{K} * g)(x) := \int_{\mathbb{R}^N} \mathcal{K}(x-z)\,g(z)\,dz \quad \text{for all } x \in \mathbb{R}^N, \tag{2.3}$$

where $\mathcal{K}$ is the Bessel kernel given by $\mathcal{K}(\xi) := \mathscr{F}^{-1}\big(\frac{1}{m+|\xi|^{2s}}\big)$ for all $\xi \in \mathbb{R}^N$. Consider the Hilbert space

$$H := \left\{ \widetilde{\varphi} \in H^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) \mid \|\widetilde{\varphi}\|_H^2 := b_s \iint_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla\widetilde{\varphi}|^2 dx\,dt + \int_{\mathbb{R}^N} m|\varphi|^2 dx < \infty \right\}. \tag{2.4}$$

From the weak form of the characterization of the fractional Laplacian given in (2.2), the solution ψ given by (2.3) can be described by the relation $\psi(x) = \widetilde{\psi}(x,0)$ in the trace sense, where $\widetilde{\psi} \in H$ is the unique solution of

$$b_s \iint_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\widetilde{\psi}\,\nabla\widetilde{\varphi}\,dx\,dt + m\int_{\mathbb{R}^N} \psi\varphi\,dx = \int_{\mathbb{R}^N} g\varphi\,dx \quad \text{for all } \varphi \in H. \tag{2.5}$$

Therefore, representations (2.3) and (2.5) are equivalent for all $g \in L^2(\mathbb{R}^N)$, see [9] for details.

2.2. **The ansatz.** To choose proper points in the definition of the proposed solutions, we start by considering points $q_i \in \omega_\varepsilon := \varepsilon^{-1}\omega$, where ω is the set given in (V_2), or equivalently, $q_i := \varepsilon^{-1}Q_i \in \mathbb{R}^N$, with $Q_i \in \omega$. Let us consider now the region Λ_ε defined by

$$\Lambda_\varepsilon := \big\{ \mathbf{q} = (q_1,\ldots,q_\ell) \in \omega_\varepsilon^\ell \mid \max_{i\neq j} |q_i - q_j| > \kappa^{-1}, \ \max_{i=1,\ldots,\ell} |q_i| < \varepsilon^{-1}\varsigma \big\} \tag{2.6}$$

for some $0 < \kappa \ll 1$ and $\varsigma \geq 1$ given.

For notational simplicity, we introduce the functions

$$w_i(x) := \lambda_i^{\frac{1}{p-1}} w(\lambda_i^{\frac{1}{2s}}(x-q_i)) \quad \text{and} \quad W(x) := \sum_{i=1}^{\ell} \tau_i w_i(x) \quad \text{for all } x \in \mathbb{R}^N, \tag{2.7}$$

where $\tau_i \in \{-1,+1\}$, $q_i \in \Lambda_\varepsilon$ and $\lambda_i = V(Q_i)$.

For $i=1,\dots,\ell$ and $l=1,\dots,N$, we introduce the functions $Z_{il}(x) := \frac{\partial w_i}{\partial x_l}(x)$ for all $x \in \mathbb{R}^N$. Clearly, Z_{il} are linearly independent, and each one belongs to $H^{2s}(\mathbb{R}^N)$ and solves the equation $L_0^i(\vartheta) := (-\Delta)^s\vartheta + V(Q_i)\vartheta - f'(w_i)\vartheta = 0 \quad \text{in } \mathbb{R}^N$. Thus, it is convenient to consider the space

$$\mathcal{Z} := \text{span}\,\{Z_{il}\}_{i=1,\dots,\ell;\, l=1,\dots,N}\,. \tag{2.8}$$

The nondegeneracy result in [14] implies that $\|\phi\|_{H^{2s}(\mathbb{R}^N)} \le c\sum_{i=1}^{\ell}\|L_0^i(\phi)\|_{L^2(\mathbb{R}^N)}$ for all $\phi \in \mathcal{Z}^\perp$ with an independent constant $c > 0$. Additionally, for functions $\psi, \varphi \in L^2(\mathbb{R}^N)$, we denote

$$\langle \psi, \varphi\rangle := \int_{\mathbb{R}^N} \psi\varphi\, dx.$$

As already explained in the introduction, solving equation (1.1) is equivalent to solving the equation (1.8). We expect to find solutions to (1.8) of the form $W + \phi$, where W is given in (2.7) and ϕ goes to 0 in $H^{2s}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Thus, we consider the problem of finding a function $\phi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} (-\Delta)^s(W+\phi) + V(\varepsilon x)(W+\phi) - f(W+\phi) = \displaystyle\sum_{i=1}^{\ell}\sum_{l=1}^{N} c_{il}Z_{il} & \text{in } \mathbb{R}^N, \\ \langle Z_{il}, \phi\rangle = 0 & \text{for all } i,l, \end{cases} \tag{2.9}$$

for certain constants c_{il} depending only on $\mathbf{q} = (q_1,\dots,q_\ell) \in \Lambda_\varepsilon$. Note that $W+\phi$ is a solution of (1.8) if all the scalars c_{il} in (2.9) are zero. Moreover, observe that (2.9) is equivalent to

$$\begin{cases} L_\varepsilon(\phi) = N_\varepsilon(\phi) + E_\varepsilon + \displaystyle\sum_{i=1}^{\ell}\sum_{l=1}^{N} c_{il}Z_{il} & \text{in } \mathbb{R}^N, \\ \langle Z_{il}, \phi\rangle = 0 & \text{for all } i,l, \end{cases} \tag{2.10}$$

where

$$L_\varepsilon(\phi) := (-\Delta)^s\phi + V(\varepsilon x)\phi - f'(W)\phi, \tag{2.11}$$

$$N_\varepsilon(\phi) := f\,(W+\phi) - f(W) - f'(W)\phi \tag{2.12}$$

and

$$E_\varepsilon := \sum_{i=1}^{\ell}\tau_i(V(Q_i) - V(\varepsilon x))w_i + f\,(W) - \sum_{i=1}^{\ell}\tau_i f(w_i). \tag{2.13}$$

2.3. **A linear problem.** We begin this subsection by introducing an appropriate L^∞-norm with weights. Specifically, for a function $h \in L^\infty(\mathbb{R}^N)$, we consider the norm $\|\cdot\|_*$ defined by

$$\|h\|_* := \|\varrho^{-\mu}h\|_{L^\infty(\mathbb{R}^N)}, \tag{2.14}$$

where

$$\varrho(x) := \sum_{i=1}^{\ell}\left(\frac{1}{1+|x-q_i|^2}\right)^{\frac{N-2s}{2}} \tag{2.15}$$

for certain points $q_i \in \mathbb{R}^N$, with $i = 1,\dots,\ell$, and $\mu \in \left(\frac{N}{2(N-2s)}, \frac{N+2s}{N-2s}\right)$.

The first step in solving (2.9) consists of addressing the following problem: given $h \in C(\mathbb{R}^N)$ verifying $\|h\|_* < \infty$, to find a function $\phi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for certain constants c_{il}, it satisfies

$$\begin{cases} L_\varepsilon(\phi) = h + \sum_{i=1}^{\ell}\sum_{l=1}^{N} c_{il} Z_{il} & \text{in } \mathbb{R}^N, \\ \langle Z_{il}, \phi\rangle = 0 & \text{for all } i, l. \end{cases} \tag{2.16}$$

Now, we shall proceed to study the invertibility of the linear operator L_ε defined in (2.11) and also to study its differentiability in terms of the variables $\mathbf{q} \in \Lambda_\varepsilon$, where Λ_ε is defined by (2.6). Consider the Banach space $\mathcal{C}^* := \{h \in C(\mathbb{R}^N) \,|\, \|h\|_* < \infty\}$, the Hilbert space H defined in (2.4) with $m = V(Q_0)$, and the Hilbert space

$$H_\varepsilon := \{\widetilde{\phi} \in H : \langle Z_{il}, \phi\rangle = 0 \quad \forall i, l\},$$

endowed with the inner product

$$[\widetilde{\phi}, \widetilde{\psi}] := b_s \iint_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\widetilde{\phi}\,\nabla\widetilde{\psi}\, dx + \int_{\mathbb{R}^N} V(\varepsilon x)\phi\psi\, dx \quad \text{for all } \widetilde{\psi} \in H_\varepsilon,$$

which is equivalent to the inner product of H. We obtain the following result of existence and uniqueness for solutions of (2.16).

Proposition 2.1. *There are numbers $M_0 > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if $\mathbf{q} \in \omega_\varepsilon^\ell$ verifies $\max_{1\le i\le \ell} |q_i| < \varepsilon^{-1}\delta_0$, $R := \min_{i\neq j} |q_i - q_j| > M_0$, then for all $\varepsilon \in (0, \varepsilon_0)$ and all $h \in \mathcal{C}^*$, problem (2.16) admits a unique solution $\phi := T_\varepsilon(h)$. Moreover, there is $C > 0$ such that $\|\phi\|_* \le C\|h\|_*$ and $\|c_{il}\|_* \le C\|h\|_*$ for all i, l.*

Henceforth, let $M_0 > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ given by Proposition 2.1, and consider the set Λ_ε for $\varepsilon \in (0, \varepsilon_0)$. Next result shows that the map $S_\varepsilon : \Lambda_\varepsilon \to \mathcal{L}(\mathcal{C}^*)$ given by $S_\varepsilon(\mathbf{q})(h) = T_\varepsilon(h)$ for all $h \in \mathcal{C}^*$, is of class C^1.

Proposition 2.2. *Under assumptions of Proposition 2.1, for each $h \in \mathcal{C}^*$, the map $\mathbf{q} \mapsto S_\varepsilon(\mathbf{q})$ is of class C^1. Moreover, there exists a constant $C > 0$ such that $\|\nabla_{\mathbf{q}}\phi\|_* \le C\|h\|_*$ uniformly on vectors $\mathbf{q} \in \Lambda_\varepsilon$, where $\phi := T_\varepsilon(h)$.*

We omit the proofs of Proposition 2.1 and Proposition 2.2 since they are slight modifications of some results in [10, 9].

2.4. **The finite-dimensional reduction.** Consider $N_\varepsilon(\phi)$ as in (2.12) and E_ε as in (2.13). After straightforward calculations, we estimate the $\|\cdot\|_*$-norm of $N_\varepsilon(\phi)$, E_ε and their respective gradients. See Appendix A for the proofs of the following two results.

Lemma 2.3. *Let $\|\phi\|_* < \frac{1}{2}$. The following estimates hold:*

$$\|N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p,2\}}, \quad \|N_\varepsilon'(\phi)\|_* \le C\|\phi\|_*^{\min\{p-1,1\}} \tag{2.17}$$

and

$$\|\nabla_{\mathbf{q}} N_\varepsilon(\phi)\|_* \le C\big(\|\phi\|_*^{\min\{p,2\}} + \|\phi\|_*^{\min\{p-1,1\}}\|\nabla_{\mathbf{q}}\phi\|_*\big). \tag{2.18}$$

Lemma 2.4. *Let $\sigma \in \big(\frac{N}{2}, \frac{N+2}{2}\big)$. For every $\mathbf{q} \in \Lambda_\varepsilon$ such that $\min_{i\neq j}|q_i - q_j| = \frac{1}{\kappa}$, the following estimates hold:*

$$\|E_\varepsilon\|_* = O\big(\varepsilon^{\min\{N+2s-\sigma,1\}} + \kappa^{N+2s-\sigma}\big) \text{ and } \|\nabla_{\mathbf{q}} E_\varepsilon\|_* = O\big(\varepsilon + \kappa^{N+2s-\sigma}\big). \tag{2.19}$$

The previous estimates will allow us to prove the existence of a unique solution ϕ of (2.10), which depends on $\mathbf{q} \in \Lambda_\varepsilon$, and certain properties of the map $\mathbf{q} \mapsto \phi = \phi(\mathbf{q})$. The proof is based on a fixed point argument, and it is given in the Appendix A.

Proposition 2.5. *Let* $\mathbf{q} \in \Lambda_\varepsilon$ *be such that* $\min_{i\neq j}|q_i - q_j| = \frac{1}{\kappa}$*, and let* $\sigma \in \left(\frac{N}{2}, \frac{N+2s}{2}\right)$*. Then, there exists* $C > 0$ *such that for all sufficiently small* ε*, a unique solution* $\phi = \phi(\mathbf{q})$ *to problem (2.10) exists. Moreover, the map* $\mathbf{q} \mapsto \phi(\mathbf{q})$ *is of* C^1*-class for the* $\|\cdot\|_*$*-norm and satisfies*

$$\|\phi\|_* \le C\big(\varepsilon^{\min\{N+2s-\sigma,1\}} + \kappa^{N+2s-\sigma}\big) \text{ and } \|\nabla_{\mathbf{q}}\phi\|_* \le C\big(\varepsilon^{\min\{N+2s-\sigma,1\}} + \kappa^{N+2s-\sigma}\big).$$

2.5. **The variational reduction.** Observe that u_ε defined in (1.7) is a solution to (1.1) if it corresponds to a stationary point of the associated energy functional $\mathcal{E}_\varepsilon$ defined formally by

$$\mathcal{E}_\varepsilon(\widetilde{u}) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^{2s}u(-\Delta)^s u + V(y)u^2\right) dy - \int_{\mathbb{R}^N} F(u)\, dy,$$

where $\widetilde{u}$ is the unique s-harmonic extension of u. Hence, letting $v_\varepsilon(x) := u_\varepsilon(\varepsilon x)$, it is sufficient to study the functional J_ε defined by

$$J_\varepsilon(\widetilde{v}) = \frac{1}{2}\int_{\mathbb{R}^N} \left(v(-\Delta)^s v + V(\varepsilon x)v^2\right) dx - \int_{\mathbb{R}^N} F(v)\, dx, \tag{2.20}$$

where $\widetilde{v}$ is the unique s-harmonic extension of v. To choose proper points in the definition of the proposed solutions, we consider points $q_i \in \Lambda_\varepsilon$, where Λ_ε is the set defined in (2.6) and the function $\phi = \phi(\mathbf{q})$ given by Proposition 2.5, which is the unique solution to (2.10). According the previous section, notice that $c_{il} = 0$ in (2.10), for all $i = 1, 2, \dots, \ell$, $l = 1, 2, \dots, N$, is equivalent to saying that

$$v_\varepsilon(x) := W(x) + \phi(x), \quad x \in \mathbb{R}^N, \tag{2.21}$$

where W is defined by (2.7), is a solution of

$$(-\Delta)^s v + V(\varepsilon x)v - f(v) = 0 \quad \text{in } \mathbb{R}^N, \tag{2.22}$$

which in turn is equivalent to saying that $u_\varepsilon(y) := W(\varepsilon^{-1}y) + \phi(\varepsilon^{-1}y)$, $y \in \mathbb{R}^N$, is a solution of (1.1). Therefore, we need to find points $\mathbf{q}$ such that the system $c_{il}(\mathbf{q}) = 0$ for all i, l has a solution. This system turns out to be equivalent to a variational problem. Precisely, consider

$$\mathcal{J}_\varepsilon(\mathbf{q}) := J_\varepsilon(\widetilde{v}_\varepsilon), \tag{2.23}$$

where $v_\varepsilon = W + \phi$ and J_ε is the functional given in (2.20).

Lemma 2.6. *The function* $u_\varepsilon(y) := W(\varepsilon^{-1}y) + \phi(\varepsilon^{-1}y)$, $y \in \mathbb{R}^N$*, is a solution of (1.1) if and only if* $\mathbf{q}$ *is a critical point of* $\mathcal{J}_\varepsilon$*.*

The proof is standard and we shall postpone to the Appendix B.

3. Proof of Theorem 1.1

In this section we work under the hypothesis of Theorem 1.1. Also we suppose that there exists a solution u_ε of the equation (1.1) having the form (1.6). It is equivalent to say that the function $v_\varepsilon = W + \phi$ given in (2.21) solves (2.22), where W is given in (2.7) and ϕ is given in Proposition 2.5. In this way, the proof of Theorem 1.1 consists in achieving a contradiction.

By convenience, in what follows, we may fix $\bar{R} > 0$ such that

$$\frac{\partial w}{\partial y_j}(z) = \frac{\bar{\gamma}(1 + o(1))}{|z|^{N+2s}} \quad \text{for all } |z| > \bar{R}, \text{ for all } j = 1, \dots, N.$$

where $o(1) \to 0$ as $|z| \to \infty$ for some $\bar{\gamma} > 0$, thanks to [14, Lemma C.2].

Lemma 3.1. *For every $l \in \{1, \dots, \ell\}$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ one has*

$$\int_{\mathbb{R}^N} f'(w_i) w_l \frac{\partial w_i}{\partial y_j}\, dz = \tilde{\gamma}_{il} \frac{(1 + o(1))}{|q_i - q_l|^{N+2s}} \frac{q_{ij} - q_{lj}}{|q_i - q_l|}, \tag{3.1}$$

for all $i = 1, \dots, \ell$; for all $j = 1, \dots, N$; where

$$\tilde{\gamma}_{il} = \bar{\gamma} V(Q_i)^{\frac{p}{p-1} - \frac{N}{2s}} V(Q_l)^{\frac{1}{p-1} - \frac{N}{2s}} \left(\int_{\mathbb{R}^N} f(w)\, dx \right) > 0.$$

Proof. To fix ideas, we consider only the case $i = 1$. Observe that $w(z) = w(|z|)$ for all z and

$$f(w(z)) \frac{w((1 + o(1))z + \xi)}{w(\xi)} \le C f(w(z)) \text{ for all } z, \xi \in \mathbb{R}^N, \text{ with } |\xi| \text{ sufficiently large.}$$

Also note that,

$$\lim_{\rho \to +\infty} \frac{w(z + \rho \mathbf{e}_j)}{w(\rho)} = \lim_{\rho \to +\infty} \left(\frac{\rho}{|(z_1, \dots, z_j, \dots, z_N) + (0, \dots, \rho, \dots, 0)|} \right)^{N+2s} = 1,$$

where $w(\rho) = w(y)$ for any y such that $|y| = \rho$. Similarly, $\frac{w'(z+\rho\mathbf{e}_j)}{w'(\rho)} \to 1$ as $\rho \to +\infty$. Hence, if we put

$$\vartheta(\rho) = \int_{\mathbb{R}^N} f(w) w \left(\frac{V(Q_l)^{\frac{1}{2s}}}{V(Q_1)^{\frac{1}{2s}}} z + \rho V(Q_l)^{\frac{1}{2s}} \mathbf{e}_j \right) dz,$$

then by Dominated Convergence Theorem we get

$$\frac{\vartheta(\rho)}{w(\rho V(Q_l)^{\frac{1}{2s}})} \to \int_{\mathbb{R}^N} f(w)\, dz \quad \text{as } \rho \to +\infty.$$

Also we have

$$\vartheta'(\rho) = V(Q_l)^{\frac{1}{2s}} \int_{\mathbb{R}^N} f(w) w' \left(\frac{V(Q_l)^{\frac{1}{2s}}}{V(Q_1)^{\frac{1}{2s}}} z + \rho V(Q_l)^{\frac{1}{2s}} \mathbf{e}_j \right) \frac{z_j + \rho V(Q_1)^{\frac{1}{2s}}}{|z + \rho V(Q_1)^{\frac{1}{2s}} \mathbf{e}_j|}\, dz,$$

that again by the Dominated Convergence Theorem leads to

$$\frac{\vartheta'(\rho)}{w'(\rho V(Q_l)^{\frac{1}{2s}})} \to V(Q_l)^{\frac{1}{2s}} \int_{\mathbb{R}^N} f(w)\, dz \quad \text{as } \rho \to +\infty.$$

Since

$$\int_{\mathbb{R}^N} f(w_1) w_l\, dz = V(Q_1)^{\frac{p}{p-1} - \frac{N}{2s}} V(Q_l)^{\frac{1}{p-1}} \vartheta(|q_1 - q_l|),$$

it follows that

$$\begin{aligned} &\frac{\partial}{\partial q_{1j}} \left(\int_{\mathbb{R}^N} f(w_1) w_l dy \right) \\ &= V(Q_1)^{\frac{p}{p-1} - \frac{N}{2s}} V(Q_l)^{\frac{1}{p-1}} \vartheta'(|q_1 - q_l|) \frac{q_{1j} - q_{lj}}{|q_1 - q_l|} \\ &= V(Q_1)^{\frac{p}{p-1} - \frac{N}{2s}} V(Q_l)^{\frac{1}{p-1} + \frac{1}{2s}} \left(\int_{\mathbb{R}^N} f(w)\, dz \right) w'(V(Q_l)^{\frac{1}{2s}} (q_1 - q_l)) \frac{q_{1j} - q_{lj}}{|q_1 - q_l|} (1 + o(1)). \end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial q_{1j}} \left(\int_{\mathbb{R}^N} f(w_1) w_l dz \right) = - \int_{\mathbb{R}^N} f'(w_1) w_l \frac{\partial w_1}{\partial z_j}\, dz.$$

Hence, by combining previous estimates we get (3.1) with $i = 1$. □

We now establish the following result that concerns the location of the peaks.

Lemma 3.2. *There exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and each $j \in \{1, \dots, N\}$, one has*

$$-\varepsilon \frac{\partial V}{\partial x_j}(Q_i)\gamma^* + \sum_{l \neq i} \frac{\hat{\gamma}_{il}(1 + o(1))}{|q_l - q_i|^{N+2s}} \frac{q_{lj} - q_{ij}}{|q_l - q_i|} + O(\varepsilon^2) + o(\kappa^{N+2s}) = 0,$$

where

$$\hat{\gamma}_{ij} = \bar{\gamma} V(Q_i)^{\frac{p-2}{p-1}} V(Q_l)^{\frac{1}{p-1} - \frac{N}{2s}} \quad \textit{and} \quad \gamma^* = \frac{1}{N}\int_{\mathbb{R}^N} ww'|y|dy < 0,$$

for all $i = 1, \dots, \ell$; for all $j = 1, \dots, N$.

Proof. Let us fix $\varepsilon_0 > 0$ sufficiently small such that $\lambda_i^{-\frac{1}{2s}} r_i = \bar{\delta}\varepsilon^{-1} > \bar{R}$ for all $\varepsilon \in (0, \varepsilon_0)$, where $\bar{\delta} > 0$ is such that the Taylor expansion of V around the point Q_0 holds. Consider $B_i^0 = \{x \in \mathbb{R}^N \,|\, |x| < r_i\}$, and $B_{\lambda_i}^q = \{x \in \mathbb{R}^N \,|\, \lambda_i^{\frac{1}{2s}}|x - q| < r_i\}$, where $\lambda_i = V(Q_i)$. It is sufficient to consider only the case when $i = 1$. Since we are assuming that $v_\varepsilon = W + \phi$ solves the equation

$$(-\Delta)^s v_\varepsilon + V(\varepsilon y)v_\varepsilon - f(v_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N.$$

Hence, multiplying the previous equality by $\frac{\partial w_1}{\partial y_j}$, by making a rearrangement of the terms and integrating on $\mathbb{R}^N$, we get

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \sum_{l=1}^{\ell} \big((-\Delta)^s w_l + V(Q_l)w_l - f(w_l)\big)\frac{\partial w_1}{\partial y_j} dy \\
&+ \int_{\mathbb{R}^N} \sum_{l=1}^{\ell} \big(V(\varepsilon y) - V(Q_l)\big)w_l \frac{\partial w_1}{\partial y_j} dy \\
&- \int_{\mathbb{R}^N} \left(f\left(\sum_{l=1}^{\ell} w_l\right) - \sum_{l=1}^{\ell} f(w_l)\right)\frac{\partial w_1}{\partial y_j} dy \\
&+ \int_{\mathbb{R}^N} \left((-\Delta)^s \phi + V(\varepsilon y)\phi - f'\left(\sum_{l=1}^{\ell} w_l\right)\phi\right)\frac{\partial w_1}{\partial y_j} dy \\
&- \int_{\mathbb{R}^N} \left(f\left(\sum_{l=1}^{\ell} w_l + \phi\right) - f\left(\sum_{l=1}^{\ell} w_l\right) - f'\left(\sum_{l=1}^{\ell} w_l\right)\phi\right)\frac{\partial w_1}{\partial y_j} dy \\
&= A_1 + A_2 - A_3 + A_4 - A_5.
\end{aligned} \tag{3.2}$$

Now we estimate every term A_i.

<u>*Estimate for A_1*</u>. Since $(-\Delta)^s w_l + V(Q_l)w_l - f(w_l) = 0$ in $\mathbb{R}^N$ for all l, then $A_1 = 0$.

<u>*Estimate for A_2*</u>. Observe that

$$\begin{aligned}
A_2 &= \int_{\mathbb{R}^N} (V(\varepsilon y) - V(Q_1))w_1 \frac{\partial w_1}{\partial y_j} dy + \int_{\mathbb{R}^N} \sum_{l \neq 1} (V(\varepsilon y) - V(Q_0))w_l \frac{\partial w_1}{\partial y_j} dy \\
&+ \int_{\mathbb{R}^N} \sum_{l \neq 1} (V(Q_0) - V(Q_l))w_l \frac{\partial w_1}{\partial y_j} dy \\
&= A_{2,1} + A_{2,2} + A_{3,3}.
\end{aligned}$$

For estimating $A_{2,1}$, we first work in $B^{q_1}_{\lambda_1}$. One has

$$\begin{aligned}
&\int_{B^{q_1}_{\lambda_1}} (V(\varepsilon x) - V(Q_1)) w_1 \frac{\partial w_1}{\partial y_j} dy \\
&= V(Q_1)^{\frac{2}{p-1}+\frac{1}{2s}} \int_{B^0_{\lambda_1}} (V(\varepsilon z + Q_1) - V(Q_1)) w\big(\lambda_1^{\frac{1}{2s}} z\big) \frac{\partial w}{\partial z_j}\big(\lambda_1^{\frac{1}{2s}} z\big)\, dz \\
&= V(Q_1)^{\frac{2}{p-1}-\frac{N}{2s}} \int_{B^0_1} \frac{\partial V}{\partial y_j}(Q_1)\varepsilon y_j\, w \frac{\partial w}{\partial y_j}\, dy + O(\varepsilon^2) \\
&= \varepsilon \lambda_1^{\frac{2}{p-1}-\frac{N}{2s}} \frac{\partial V}{\partial y_j}(Q_1)\gamma^* + O(\varepsilon^2),
\end{aligned}$$

where

$$\gamma^* = \int_{\mathbb{R}^N} z_j w \frac{\partial w}{\partial z_j}\, dz = \int_{\mathbb{R}^N} w w' \frac{z_j^2}{|z|}\, dz = \frac{1}{N} \int_{\mathbb{R}^N} w w' |z|\, dz < 0.$$

Now we work on $\mathbb{R}^N \setminus B^{q_1}_{\lambda_1}$. Since V is bounded and

$$\Big| \int_{\mathbb{R}^N \setminus B^{q_1}_{\lambda_1}} w_1 \frac{\partial w_1}{\partial y_j} dy \Big| = O(\varepsilon^{N+4s}),$$

we get

$$\int_{\mathbb{R}^N \setminus B^{q_1}_{\lambda_1}} (V(\varepsilon y) - V(Q_1)) w_1 \frac{\partial w_1}{\partial y_j} dy = O(\varepsilon^{N+4s}).$$

Then

$$A_{2,1} = \varepsilon \lambda_1^{\frac{2}{p-1}-\frac{N}{2s}} \frac{\partial V}{\partial y_j}(Q_1)\gamma^* + O(\varepsilon^{N+4s}).$$

For $A_{2,2}$, observe first that

$$\int_{\mathbb{R}^N} w_l \frac{\partial w_1}{\partial y_j} dy = \lambda_1^{\frac{1}{p-1}} \lambda_l^{\frac{1}{p-1}-\frac{N}{2s}} \int_{\mathbb{R}^N} w(y) \frac{\partial w}{\partial y_j}\Big(\Big(\frac{\lambda_1}{\lambda_l}\Big)^{\frac{1}{2s}} y + \lambda_1^{\frac{1}{2s}}(q_l - q_1)\Big)\, dy.$$

Recalling the definition of Λ_ε, for κ sufficiently small, we have $|q_l - q_1| = O\big(\frac{1}{\kappa}\big) > R_0$ if $l \neq 1$. Then,

$$\frac{\partial w}{\partial y_j}\Big(\Big(\frac{\lambda_1}{\lambda_l}\Big)^{\frac{1}{2s}} y + \lambda_1^{\frac{1}{2s}}(q_l - q_1)\Big) = -\frac{\lambda_1^{\frac{1}{2s}}}{\lambda_l^{\frac{1}{2s}}} \frac{\bar{\gamma}(1+o(1))}{\lambda_1^{\frac{N+2s}{2s}} |q_l - q_1|^{N+2s}} \frac{q_{lj} - q_{1j}}{|q_l - q_1|}.$$

Now, note that

$$\begin{aligned}
&\int_B \Bigg(V\Bigg(\varepsilon \frac{1}{\lambda_1^{\frac{1}{2s}}} z + Q_0\Bigg) - V(Q_0)\Bigg) w(z + q_0)\, dz \\
&= \frac{1}{2} \frac{\varepsilon^2}{\lambda_1^{\frac{1}{s}}} \int_B D^2 V(Q_0)|z|^2 w(z+q_0)\, dz + o\Bigg(\varepsilon^2 \int_B D^2 V(Q_0)|z|^2 w(z+q_0)\, dz\Bigg) \\
&= O(\varepsilon^2),
\end{aligned}$$

where $B = B(0, R_0)$. Besides, since $V \in L^\infty(\mathbb{R}^N)$, and $\int_{\mathbb{R}^N} w\, dz < \infty$, by the Dominated Convergence Theorem we get

$$\int_{\mathbb{R}^N \setminus B(0,R_0)} \Bigg(V\Bigg(\varepsilon \frac{1}{\lambda_1^{\frac{1}{2s}}} z + Q_0\Bigg) - V(Q_0)\Bigg) w(z+q_0)\, dz \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows that

$$\int_{\mathbb{R}^N} \sum_{l \neq 1} (V(\varepsilon y) - V(Q_0)) w_l \frac{\partial w_1}{\partial y_j}\, dy = o\Bigg(\frac{1}{|q_1 - q_l|^{N+2s}} \frac{q_{1j} - q_{lj}}{|q_1 - q_l|}\Bigg) + O(\varepsilon^2).$$

Therefore, we get $A_{2,2} = o(\kappa^{N+2s}) + O(\varepsilon^2)$.

For $A_{2,3}$, we take into account that $V(Q_1) \to V(Q_0)$ as $\varepsilon \to 0$, from where we get immediately $A_{2,3} = o(\kappa^{N+2s})$. Thus

$$A_2 = \varepsilon V(Q_1)^{\frac{2}{p-1}-\frac{N}{2s}} \frac{\partial V}{\partial y_j}(Q_1)\,\gamma^* + O(\varepsilon^2) + o(\kappa^{N+2s}).$$

<u>*Estimate for A_3.*</u> Decompose A_3 in two terms,

$$\begin{aligned} A_3 &= \int_{B^{q_1}_{\lambda_1}} \Big(f\Big(\sum_{l=1}^{\ell} w_l\Big) - f(w_1)\Big) \frac{\partial w_1}{\partial y_j}\,dy + \int_{\mathbb{R}^N \setminus B^{q_1}_{\lambda_1}} \Big(f\Big(\sum_{l=1}^{\ell} w_l\Big) - f(w_1)\Big) \frac{\partial w_1}{\partial y_j}\,dy \\ &= A_{3,1} + A_{3,2}. \end{aligned}$$

On $B^{q_1}_{\lambda_1}$ we have

$$\begin{aligned} A_{3,1} &= \int_{B^{q_1}_{\lambda_1}} \sum_{l\neq 1} f'(w_1) w_l \frac{\partial w_1}{\partial y_j}\,dy + o\Big(\int_{B^{q_1}_{\lambda_1}} \sum_{l\neq 1} f'(w_1) w_l \frac{\partial w_1}{\partial y_j}\,dy\Big) \\ &= \sum_{l\neq 1} \frac{\tilde{\gamma}_{l1}(1+o(1))}{|q_1 - q_l|^{N+2s}} \frac{q_{1j} - q_{lj}}{|q_1 - q_l|} + o(\kappa^{N+2s}), \end{aligned}$$

where we have use Lemma 3.1. Outside $B^{q_1}_{\lambda_1}$ we get

$$A_{3,2} = o\Big(\sum_{l\neq 1} \frac{\gamma_0}{|q_l - q_1|^{N+2s}}\Big) = o(\kappa^{N+2s}),$$

that implies

$$A_3 = \sum_{l\neq 1} \frac{\tilde{\gamma}_{1l}(1+o(1))}{|q_1 - q_l|^{N+2s}} \frac{q_{1j} - q_{lj}}{|q_1 - q_l|} + o(\kappa^{N+2s}).$$

<u>*Estimate for A_4.*</u> Note that

$$(-\Delta)^s \frac{\partial w_1}{\partial y_j} + V(Q_1)\frac{\partial w_1}{\partial y_j} - f'(w_1)\frac{\partial w_1}{\partial y_j} = 0.$$

This last equality implies that

$$\begin{aligned} A_4 &= \int_{\mathbb{R}^N} \big(V(\varepsilon y) - V(Q_1)\big) \frac{\partial w_1}{\partial y_j}\phi\,dy - \int_{\mathbb{R}^N} \Big(f'\Big(\sum_{l=1}^{\ell} w_l\Big) - f'(w_1)\Big)\frac{\partial w_1}{\partial y_j}\phi\,dy \\ &= A_{4,1} + A_{4,2}. \end{aligned}$$

Observe that

$$\int_{B^{q_1}_{\lambda_1}} (V(\varepsilon y) - V(Q_1))\phi \frac{\partial w_1}{\partial y_j} dy = \varepsilon \lambda_1^{\frac{1}{p-1}-\frac{N}{2s}} \int_{B^0_{\lambda_1}} \nabla V(Q_1)\cdot y\,\phi(\lambda_1^{-\frac{1}{2s}} y + q_1)\frac{\partial w}{\partial y_j} dy + O(\varepsilon^2)$$

and

$$\int_{\mathbb{R}^N \setminus B^{q_1}_{\lambda_1}} (V(\varepsilon y) - V(Q_1))\phi \frac{\partial w_1}{\partial y_j} dy = O(\varepsilon \kappa^{N+2s}).$$

Thus, by Proposition 2.5 we deduce that

$$\begin{aligned} |A_{4,1}| &= O\Big(\varepsilon \int_{\mathbb{R}^N} |y| |\varrho(\lambda^{-\frac{1}{2s}} y + q_1))^{\mu}| \Big|\frac{\partial w}{\partial y_j}\Big|\,dy\Big) \|\phi\|_* + O(\varepsilon\kappa^{N+2s}) \\ &\leq O(\varepsilon^2 + \varepsilon\kappa^{N+2s}), \end{aligned}$$

where ϱ is the function defined in (2.15) and μ the corresponding value given in the definition of the norm $\|\cdot\|_*$ in (2.14). Besides, note that

$$\int_{\mathbb{R}^N}\Big(f'\Big(\sum_{l=1}^{\ell}w_l\Big)\sum_{l=1}^{\ell}\frac{\partial w_l}{\partial y_j}-f'(w_1)\frac{\partial w_1}{\partial y_j}\Big)\phi\,dy=\int_{\mathbb{R}^N}\frac{\partial}{\partial y_j}\Big(f\Big(\sum_{l=1}^{\ell}w_l\Big)-f(w_1)\Big)\phi\,dy$$
$$=-\int_{\mathbb{R}^N}\Big(f\Big(\sum_{l=1}^{\ell}w_l\Big)-f(w_1)\Big)\frac{\partial\phi}{\partial y_j}\,dy.$$

Hence

$$\begin{aligned}|A_{4,2}|&=O\Big(\int_{\mathbb{R}^N}|\varrho(y)^{\mu}|\Big|f\Big(\sum_{l=1}^{\ell}w_l\Big)-f(w_1)\Big|\,dy\Big)\|\nabla\phi\|_*\\&=O\Big(\int_{\mathbb{R}^N}\frac{1}{(1+|y-q_1|)^{(N+2s)(p-1)+\sigma}}dy\Big)\sum_{l\neq1}\frac{(1+o(1))}{|q_l-q_1|^{N+2s-\sigma}}\|\nabla\phi\|_*.\end{aligned}$$

The previous two equalities and the estimate of $\|\nabla\phi\|_*$ in Proposition 2.5 lead to

$$A_4=O\big(\varepsilon^{\min\{N+2s-\sigma,1\}}+\kappa^{N+2s-\sigma}\big)\sum_{l\neq1}\frac{(1+o(1))}{|q_l-q_1|^{N+2s-\sigma}}\leq o(\kappa^{N+2s}).$$

Estimate for A_5. Observe that by Lemma 2.3 one has

$$\Big|\int_{\mathbb{R}^N}N_\varepsilon(\phi)\phi\frac{\partial w_1}{\partial y_j}dy\Big|\leq\|\phi\|_*^{\min\{p+1,3\}}O\Big(\int_{\mathbb{R}^N}|\varrho(y)|^{2\mu}\Big|\frac{\partial w_1}{\partial y_j}\Big|dy\Big),$$

so that, by Proposition 2.5 we get $A_5=O\big(\varepsilon^2+\varepsilon\kappa^{N+2s}\big)$.

We finish the proof by considering the previous estimates for each A_i and replacing all of them at the equation (3.2). □

Proof of the Theorem 1.1. We have now all ingredients to finish the proof. Indeed, it is enough for us to follow closely the final argument of the proof of [15, Theorem 1.2]. Without loss of generality, we can assume that $|Q_1-Q_2|=\min_{i\neq j}|Q_i-Q_j|=d_\varepsilon$. In this way, if we put

$$\delta_{ij}^{\varepsilon}=\frac{(1+o(1))}{|q_i-q_j|^{N+2s}}=\varepsilon^{N+2s}\frac{(1+o(1))}{|Q_i-Q_j|^{N+2s}},$$

then

$$\delta_{12}^{\varepsilon}=\max_{i\neq j}\delta_{ij}^{\varepsilon}=\frac{\varepsilon^{N+2s}}{d_\varepsilon^{N+2s}}=\delta_\varepsilon.$$

For $\varepsilon>0$ given, consider the set $\mathcal{Q}_\varepsilon\subset\Lambda_\varepsilon$ of the ℓ points where the solution of (1.1) is concentrated; *i.e.* $\mathcal{Q}_\varepsilon=\{Q_1,\dots,Q_l\}$. Let

$$\mathcal{S}_\varepsilon=\Big\{\frac{Q_k}{\varepsilon}\in\mathcal{Q}_\varepsilon\,|\,Q_k=Q_1\text{ or }\exists Q_{k_1},\dots,Q_{k_l}\text{ such that }\lim_{\varepsilon\to0}\frac{|Q_{k_j}-Q_{k_1}|}{d_\varepsilon}=1,j=2,\dots,l\Big\}.$$

It is not difficult to check that there exists $Q_i\in\mathcal{S}_\varepsilon$ and a hyperplane $\mathcal{H}$, such that $Q_i\in\mathcal{H}$ and any other point of $\mathcal{S}_\varepsilon$ belongs to the same half-space of $\mathbb{R}^N$ divided by $\mathcal{H}$. If $\frac{\varepsilon}{\delta_\varepsilon}\to0$, we can divided the equation on Lemma 3.2 by δ_ε to obtain

$$\sum_{l\neq i}\frac{\delta_{il}^{\varepsilon}}{\delta_\varepsilon}\frac{(Q_{lj}-Q_{ij})}{|Q_l-Q_i|}=o(1)\quad\text{and}\quad\sum_{l\neq i,Q_i\in\mathcal{S}_\varepsilon}\frac{\delta_{il}^{\varepsilon}}{\delta_\varepsilon}\frac{Q_{lj}-Q_{ij}}{|Q_l-Q_i|}=o(1).\tag{3.3}$$

Since $Q_i\in\mathcal{S}_\varepsilon$ there exists $l\neq i$ such that $\lim_{\varepsilon\to0}\frac{\delta_{il}^{\varepsilon}}{\delta_\varepsilon}=1>0$ and all Q_j^{ε}, with $j\neq i$, belong the same half-space of $\mathbb{R}^N$ divide by $\mathcal{H}$, but this is a contradiction with (3.3). Therefore $\delta_\varepsilon=O(\varepsilon)$.

Now we choose a point $\bar{Q}\in\mathcal{Q}_\varepsilon$ such that $|\bar{Q}-Q_0|=\max_{j=1,\dots,\ell}|Q_j-Q_0|:=\alpha_\varepsilon$. Without loss of generality, we can assume that $\bar{Q}=Q_1$ and by an appropriate rotation and translation, also we can assume that $Q_0 = 0$ and $Q_1 = (-\alpha_\varepsilon, 0, \dots, 0)$. For others points Q_l, with $l \neq 1$, we have $Q_{l1} - Q_{11} > 0$. By Lemma 3.2 we have

$$-\frac{\partial V}{\partial x_1}(Q_1) + \frac{1}{\gamma^*}\sum_{l\neq 1}\frac{\hat{\gamma}_{1l}\delta^\varepsilon_{1l}}{\varepsilon}\frac{Q_{l1}-Q_{11}}{|Q_l-Q_1|} + O(\varepsilon) + o(\varepsilon^{-1}\kappa^{N+2s}) = 0.$$

Now, given $0 \le \rho < N + 2s - 2$, we can take $\kappa = \varepsilon^{\frac{2+\rho}{N+2s}}$. Notice that

$$\frac{\delta_\varepsilon}{\varepsilon} = O(\alpha_\varepsilon). \tag{3.4}$$

Indeed, if $\frac{\delta_\varepsilon}{\varepsilon} \neq O(\alpha_\varepsilon)$, then $\alpha_\varepsilon = o(\frac{\delta_\varepsilon}{\varepsilon})$ and $|Q_l| = o(\frac{\delta_\varepsilon}{\varepsilon})$. As before, there is a point $Q_i \in \mathcal{S}_\varepsilon$ and a hyperplane $\mathcal{H}$ such that $Q_i \in \mathcal{H}$ and all the other points belong to the same half-space of $\mathbb{R}^N$ divided by $\mathcal{H}$. By Lemma 3.2 we obtain

$$\frac{1}{\gamma^*}\sum_{l\neq i}\frac{\hat{\gamma}_{il}\delta^\varepsilon_{il}}{\varepsilon}\frac{Q_{lj}-Q_{ij}}{|Q_l-Q_i|} + o\Big(\frac{\delta_\varepsilon}{\varepsilon}\Big) + O(\varepsilon) + o(\varepsilon^{1+\rho}) = 0. \tag{3.5}$$

Nevertheless, there exists $l \neq i$ such that $\lim_{\varepsilon\to 0}\frac{\delta^\varepsilon_{il}}{\delta_\varepsilon} = 1$ and for all Q_l such that

$$\lim_{\varepsilon\to 0}\frac{|Q_l - Q_i|}{d_\varepsilon} = 1,$$

$Q_l - Q_i$ are vectors lying on the same half-space, which leads to a contradiction with (3.5).

Since (3.4) holds, we get

$$V_{11}(0)\alpha_\varepsilon + \frac{1}{\gamma^*}\sum_{l\neq 1}\frac{\hat{\gamma}_{1l}\delta^\varepsilon_{1l}}{\varepsilon\alpha_\varepsilon}\frac{Q_{l1}-Q_{11}}{|Q_l-Q_1|}\alpha_\varepsilon + o(\alpha_\varepsilon) = 0,$$

which is impossible due to that

$$V_{11}(0) + \frac{1}{\gamma^*}\sum_{l\neq 1}\frac{\hat{\gamma}_{1l}\delta^\varepsilon_{1l}}{\varepsilon\alpha_\varepsilon}\frac{Q_{l1}-Q_{11}}{|Q_l-Q_1|} > V_{11}(0) > 0.$$

This completes the proof. □

4. Proof of the Theorem 1.2

4.1. **The reduced energy.** Since solutions of (2.22) correspond to stationary points of its associated energy functional J_ε defined by (2.20), we have that if a solution of the form $v_\varepsilon := W + \phi$ in $\mathbb{R}^N$ exists for (2.22), in which W is defined by (2.7) and $\phi = \phi(\mathbf{q})$ is given in Proposition 2.5, then we should have $J_\varepsilon(\tilde{v}_\varepsilon) \sim J_\varepsilon(\widetilde{W})$, where $\tilde{v}_\varepsilon$ and $\widetilde{W}$ are the unique s-harmonic extensions of v_ε and W, respectively, and the corresponding points $\mathbf{q}$ in the definition of W also should be approximately stationary for the finite-dimensional functional $\mathbf{q} \mapsto J_\varepsilon(\widetilde{W})$. Thus, our next goal is to estimate $J_\varepsilon(\widetilde{W})$. The first lemma contains a crucial estimate for this aim. By convenience, in what follows, we fix $R_0 > 0$ such that

$$w(z) = \frac{\gamma_0(1+o(1))}{|z|^{N+2s}} \quad \text{for all } |z| > R_0, \text{ for all } j = 1, \dots, N.$$

where $o(1) \to 0$ as $|z| \to \infty$ for some $\gamma_0 > 0$, thanks to (1.3).

Lemma 4.1. *Let $\sigma \in \big(\frac{N}{2}, N+2s\big)$ be fixed. For any $\mathbf{q} = (q_1, \dots, q_\ell) \in \Lambda_\varepsilon$ such that $\min_{i\neq j}|q_i - q_j| \ge \frac{1}{\kappa}$, where Λ_ε is defined by (2.6), and for sufficiently small ε, we have*

$$J_\varepsilon(\widetilde{W}) = \sum_{i=1}^{\ell} V(Q_i)^{\frac{p+1}{p-1}-\frac{N}{2s}} I_1(w) - \gamma\sum_{i\neq j}\tau_i\tau_j\frac{\gamma_{ij}(1+o(1))}{|q_i-q_j|^{N+2s}}$$
$$+O\big(\varepsilon^{\min\{N+2s,2\}}\big) + o\big(\kappa^{N+2s}\big),$$

where $\tau_i \in \{-1,+1\}$, I_1 *is given by* (1.5) *with* $\lambda = 1$,

$$\gamma := \frac{\gamma_0}{2}\int_{\mathbb{R}^N} f(w)\,dx \quad \text{and} \quad \gamma_{ij} := V(Q_i)^{\frac{1}{p-1}-\frac{N+2s}{2s}} V(Q_j)^{\frac{p}{p-1}} \quad \text{for } i \neq j.$$

Proof. Let us fix $\varepsilon_0 > 0$ sufficiently small such that $\lambda_i^{-\frac{1}{2s}} r_i = \bar{\delta}\varepsilon^{-1} > R_0$ for all $\varepsilon \in (0, \varepsilon_0)$. Observe that

$$\begin{aligned} J_\varepsilon(\widetilde{W}) = &\sum_{i=1}^{\ell} J_\varepsilon(\widetilde{w}_i) + \frac{1}{2}\int_{\mathbb{R}^N}\Big(\sum_{i\neq j}\tau_i\tau_j w_i(-\Delta)^s w_j\Big)dx \\ &+\frac{1}{2}\int_{\mathbb{R}^N}\Big(\sum_{i\neq j} V(\varepsilon x)\tau_i\tau_j w_i w_j\Big)dx - \int_{\mathbb{R}^N}\Big(F(W) - \sum_{i=1}^{\ell} F(w_i)\Big)dx. \end{aligned}$$

It is easy to check that

$$J_\varepsilon(\widetilde{w}_i) = V(Q_i)^{\frac{p+1}{p-1}-\frac{N}{2s}} I_1(w) + \frac{1}{2}\int_{\mathbb{R}^N}(V(\varepsilon x) - V(Q_i))w_i^2(x)\,dx.$$

Moreover, since $\int_{B_i^0} z_i w^2(z)\,dz = 0$ for all i, where $B_i^0 = \{x \in \mathbb{R}^N \mid |x| < r_i\}$, by letting $B_{\lambda_i}^q = \{x \in \mathbb{R}^N \mid \lambda_i^{\frac{1}{2s}}|x-q| < r_i\}$, we obtain

$$\begin{aligned} \int_{B_{\lambda_i}^{q_i}} (V(\varepsilon x) - V(Q_i))w_i^2\,dx &= \lambda_i^{\frac{2}{p-1}}\int_{B_{\lambda_i}^0}(V(Q_i+\varepsilon z) - V(Q_i))w^2\big(\lambda_i^{\frac{1}{2s}} z\big)\,dz \\ &= \lambda_i^{\frac{2}{p-1}-\frac{N}{2s}-\frac{1}{2s}}\int_{B_i^0}\nabla V(Q_i)\cdot \varepsilon z\, w^2\,dz + O(\varepsilon^2) \\ &= O(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N\setminus B_{\lambda_i}^{q_i}}(V(\varepsilon x) - V(Q_i))w_i^2\,dx &= \lambda_i^{\frac{2}{p-1}}\int_{\mathbb{R}^N\setminus B_{\lambda_i}^0}(V(Q_i+\varepsilon z) - V(Q_i))w^2\big(\lambda_i^{\frac{1}{2s}} z\big)\,dz \\ &\leq C\varepsilon^{2(N+2s-\sigma)}\int_{r_i}^{\infty}\frac{\rho^{N-1}}{\rho^{2\sigma}}d\rho \\ &= O\big(\varepsilon^{N+2s}\big). \end{aligned}$$

On the other hand, using the equation solved by w_j, $i \neq j$, we obtain

$$\int_{\mathbb{R}^N}\tau_i\tau_j(w_i(-\Delta)^s w_j + \lambda_j w_i w_j - w_i f(w_j))\,dx = 0.$$

Now, we estimate each term of interactions between two different peaks. We have

$$\int_{\mathbb{R}^N} w_i f(w_j)\,dx = \lambda_i^{\frac{1}{p-1}}\lambda_j^{\frac{p}{p-1}-\frac{N}{2s}}\int_{\mathbb{R}^N} f(w(y))w\Big(\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\frac{1}{2s}} y + \lambda_i^{\frac{1}{2s}}(q_j - q_i)\Big)\,dy.$$

Recalling the definition of Λ_ε, for κ sufficiently small, we have that $|q_i - q_j| > R_0$ if $i \neq j$. Then,

$$w\Big(\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\frac{1}{2s}} y + \lambda_i^{\frac{1}{2s}}(q_j - q_i)\Big) = \frac{\gamma_0(1+o(1))}{\lambda_i^{\frac{N+2s}{2s}}|q_j - q_i|^{N+2s}}.$$

Therefore, we obtain

$$\int_{\mathbb{R}^N} w_i f(w_j)\,dx = \lambda_i^{\frac{1}{p-1}-\frac{N+2s}{2s}}\lambda_j^{\frac{p}{p-1}-\frac{N}{2s}}\Big(\int_{\mathbb{R}^N} f(w)\,dx\Big)\frac{\gamma_0}{|q_j - q_i|^{N+2s}}(1+o(1)).$$

Now, we estimate

$$\int_{\mathbb{R}^N}(w_i(-\Delta)^s w_j + V(\varepsilon x)w_i w_j)\,dx = \int_{\mathbb{R}^N}(w_i f(w_j) + (V(\varepsilon x) - V(Q_j))w_i w_j)\,dx.$$

The first term has been analyzed. Let us continue with the second term. We have

$$\int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \, dx = \lambda_i^{\frac{1}{p-1}} \lambda_j^{\frac{1}{p-1}} \int_{\mathbb{R}^N} V(\varepsilon x) w\big(\lambda_i^{\frac{1}{2s}}(x-q_i)\big) w\big(\lambda_j^{\frac{1}{2s}}(x-q_j)\big)\, dx.$$

Also we have

$$\left| \int_{B^{q_i}_{\lambda_i}} (V(\varepsilon x) - V(Q_i)) w_i w_j \, dx \right| = O(\varepsilon^2)$$

and

$$\left| \int_{\mathbb{R}^N \setminus B^{q_i}_{\lambda_i}} (V(\varepsilon x) - V(Q_i)) w_i w_j \, dx \right| = o\left(\kappa^{N+2s}\right).$$

Finally, letting $\Omega_i = \{x \in \Omega : w_i > w_j \text{ for all } j \neq i\}$, we obtain

$$\begin{aligned}
&\int_{\Omega_i} \left(F(W) - \sum_{i=1}^{\ell} F(w_i) \right) dx \\
&= \sum_{j \neq i} \int_{\Omega_i} \tau_i \tau_j f(w_i) w_j \, dx - \sum_{j \neq i} \int_{\Omega_i} F(w_j)\, dx + O\Big(\sum_{j\neq i} \int_{\Omega_i} f'(w_i) w_j^2 \, dx \Big) \\
&= \sum_{i \neq j} \tau_i \tau_j \lambda_i^{\frac{p}{p-1} - \frac{N}{2s}} \lambda_j^{\frac{1}{p-1} - \frac{N+2s}{2s}} \frac{\gamma_0 (1 + o(1))}{|q_i - q_j|^{N+2s}} \int_{\mathbb{R}^N} f(w)\, dx + O\big(\varepsilon^{2\sigma - N} \kappa^{2(N+2s-\sigma)}\big).
\end{aligned}$$

Thus, the proof is completed considering all previous estimates. □

4.2. **A suitable expansion of $\mathcal{J}_\varepsilon$.** We start this subsection by validating an expansion for the functional $\mathcal{J}_\varepsilon$ given in (2.23) for $\mathbf{q} \in \Lambda_\varepsilon$, the set defined in (2.6), which will be crucial for finding its critical points. Here, W is given by (2.7).

Proposition 4.2. *Let $\mathbf{q} \in \Lambda_\varepsilon$ be such that $\min_{i\neq j} |q_i - q_j| = \frac{1}{\kappa}$, and let $\sigma \in \left(\frac{N}{2}, \frac{N+2s}{2}\right)$. If $\phi = \phi(\mathbf{q})$ is the function given in Proposition 2.5, then the following expansions hold:*

$$\mathcal{J}_\varepsilon(\mathbf{q}) = J_\varepsilon\big(\widetilde{W}\big) + O\big(\varepsilon^{\min\{2(N+2s-\sigma),2\}} + \kappa^{2(N+2s-\sigma)}\big)$$

and

$$\nabla_{\mathbf{q}} \mathcal{J}_\varepsilon(\mathbf{q}) = \nabla_{\mathbf{q}} J_\varepsilon\big(\widetilde{W}\big) + O\big(\varepsilon^{\min\{2(N+2s-\sigma),2\}} + \kappa^{2(N+2s-\sigma)}\big)$$

uniformly on the vectors $\mathbf{q}$.

Proof. Let $\mathbf{q} \in \overline{\Lambda}_\varepsilon$. Then,

$$\begin{aligned}
\mathcal{J}_\varepsilon(\mathbf{q}) \;=\; & J_\varepsilon\big(\widetilde{W}\big) - \frac{1}{2} \int_{\mathbb{R}^N} \phi E_\varepsilon \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \phi (f(W+\phi) - f(W))\, dx \\
& - \int_{\mathbb{R}^N} \big(F(W+\phi) - F(W) - f(W)\phi \big)\, dx \\
& + \frac{1}{2} \int_{\mathbb{R}^N} \phi ((-\Delta)^s (W+\phi) + V(\varepsilon x)(W+\phi) - f(W+\phi)\, dx \\
& + \frac{1}{2} \sum_{i=1}^{\ell} \tau_i \int_{\mathbb{R}^N} \phi ((-\Delta)^s w_i + V(Q_i) w_i - f(w_i))\, dx.
\end{aligned}$$

Since $(-\Delta)^s (W + \phi + V(\varepsilon x)(W+\phi) - f(W+\phi) \in \mathcal{Z}$, with $\mathcal{Z}$ defined by (2.8), and ϕ is L^2- orthogonal to $\mathcal{Z}$, we obtain

$$\int_{\mathbb{R}^N} \phi ((-\Delta)^s (W+\phi) + V(\varepsilon x)(W+\phi) - f(W+\phi))\, dx = 0.$$

Besides, $(-\Delta)^s w_i + V(Q_i) w_i - f(w_i) = 0 \quad \text{in } \mathbb{R}^N$ for all $i = 1, \dots, \ell$. Now, observe that, because of (2.19) and Proposition 2.5, we have

$$\left| \int_{\mathbb{R}^N} (f(W+\phi) - f(W)) - E_\varepsilon)\phi \, dx \right| = O\big(\varepsilon^{\min\{2(N+2s-\sigma),2\}} + \kappa^{2(N+2s-\sigma)}\big)$$

and

$$\left|\int_{\mathbb{R}^N}\big(F(W+\phi)-F(W)-f(W)\phi\big)\,dx\right| = O\big(\varepsilon^{\min\{2(N+2s-\sigma),2\}}+\kappa^{2(N+2s-\sigma)}\big).$$

Then, considering estimates in Lemma 4.1, the desired estimate for $\mathcal{J}_\varepsilon(\mathbf{q})$ follows. Now, observe that

$$\begin{aligned}\nabla_{\mathbf{q}}\mathcal{J}_\varepsilon(\mathbf{q}) = \nabla_{\mathbf{q}}J_\varepsilon(\widetilde{W}) &- \frac{1}{2}\int_{\mathbb{R}^N}\nabla_{\mathbf{q}}\phi E_\varepsilon\,dx - \frac{1}{2}\int_{\mathbb{R}^N}\phi\nabla_{\mathbf{q}}E_\varepsilon\,dx\\ &+\frac{1}{2}\int_{\mathbb{R}^N}\nabla_{\mathbf{q}}\phi(f(W+\phi)-f(W))dx+\frac{1}{2}\int_{\mathbb{R}^N}(f'(W+\phi)-f'(W))\phi\,\nabla_{\mathbf{q}}W\,dx\\ &+\frac{1}{2}\int_{\mathbb{R}^N}f'(W+\phi)(\nabla_{\mathbf{q}}\phi)\phi\,dx-\int_{\mathbb{R}^N}\big(f(W+\phi)-f(W)-f'(W)\phi\big)\nabla_{\mathbf{q}}W\,dx\\ &-\int_{\mathbb{R}^N}\big(f(W+\phi)-f(W)\big)\nabla_{\mathbf{q}}\phi\,dx.\end{aligned}$$

Then, by using the estimates for $\|\phi\|_*$ and $\|\nabla_{\mathbf{q}}\phi\|_*$ in Proposition 2.5 and for $\|E_\varepsilon\|_*$ and $\|\nabla_{\mathbf{q}}E_\varepsilon\|_*$ given in (2.19), we are able to prove the estimate for $\nabla_{\mathbf{q}}\mathcal{J}_\varepsilon(\mathbf{q})$. Therefore, the proof is completed. □

4.3. **A minimization argument.** Let $\ell\in\mathbb{N}$, and let us set $\tau_{2l-1}=1$ and $\tau_{2l}=-1$ for $l=1,\dots,\ell$. By following the arguments developed in [15, 9], we choose the configuration space:

$$\Sigma_\varepsilon=\Big\{\mathbf{q}\in\Lambda_\varepsilon\,|\,q_1,\dots,q_{2\ell}\in\omega_\varepsilon\quad\text{and}\quad\min_{i\neq j}|q_i-q_j|>\varepsilon^{-\frac{s}{N+2s}}\Big\},$$

with Λ_ε given by (2.6), with $\kappa=\varepsilon$.

Proposition 4.3. *The problem*

$$\min_{\mathbf{q}\in\Sigma_\varepsilon}\mathcal{J}_\varepsilon(\mathbf{q})$$

admits a minimizer $\mathbf{q}_\varepsilon=(q_{1,\varepsilon},\dots,q_{2\ell,\varepsilon})\in\Sigma_\varepsilon$ *for all* $\varepsilon>0$ *sufficiently small.*

Proof. Since $\mathcal{J}_\varepsilon$ is continuous, there is a minimizer $\mathbf{q}_\varepsilon=(q_{1,\varepsilon},\dots,q_{2\ell,\varepsilon})\in\overline{\Sigma}_\varepsilon$. To validate our result, we need to prove that $\mathbf{q}_\varepsilon\in\Sigma_\varepsilon$. Then, we start looking for an upper bound on $\mathcal{J}_\varepsilon(\mathbf{q}_\varepsilon)$. We choose $Q_i^0=Q_0+\varepsilon^{\frac{N}{N+2s}}X_i$, where X_i, $i=1,2,\dots,2\ell$ are the 2ℓ vortices of a regular 2ℓ-polygon centered at Q_0 with $|X_{2k+1}-X_{2k}|=1$ for all $k=1,\dots,\ell$, where we have considered $X_{2\ell+1}=X_1$, and $\min_{i\neq j}|X_i-X_j|=1$. Moreover, choose sufficiently small $\varepsilon>0$ and let $\mathbf{q}^0\in\omega_\varepsilon$ such that

$$\frac{1}{|q_{2k}^0-q_{2k+1}^0|^{N+2s}}=\varepsilon^{2s}\quad\text{for all } k=1,\dots,\ell,$$

where we have considered $q_{2\ell+1}^0=q_1^0$. Therefore, $\mathbf{q}^0=(q_1^0,\dots,q_{2\ell}^0)\in\Sigma_\varepsilon$. Note that in general, if $i\neq j$, then $|X_i-X_j|\le C_{ij}$, where $1\le C_{ij}\le C_0$, with $C_0=\max_{i\neq j}|X_i-X_j|$. Then, we obtain

$$\frac{1}{|q_i^0-q_j^0|^{N+2s}}\ge\frac{1}{C_0^{N+2s}}\varepsilon^{2s}\quad\text{for all } i\neq j,\qquad 0<-\sum_{i\neq j}\tau_i\tau_j\frac{1}{|q_i^0-q_j^0|^{N+2s}}=O\big(\varepsilon^{2s}\big),$$

and, by Taylor's expansion, $V(Q_i^0)=V(Q_0)+O\big(\varepsilon^{\frac{2N}{N+2s}}\big)$ for all $\varepsilon>0$ sufficiently small. Indeed,

$$\begin{aligned}-\sum_{i\neq j}\tau_i\tau_j\frac{1}{|q_i^0-q_j^0|^{N+2s}}&=-\sum_{j=1}^{2l-1}\sum_{i=1}^{2l-j}\tau_i\tau_{i+j}\frac{1}{|q_i^0-q_{i+j}^0|^{N+2s}}\\ &=-\sum_{k=1}^{l}\sum_{i=1}^{2l-(2k-1)}\tau_i\tau_{i+2k-1}\frac{1}{|q_i^0-q_{i+2k-1}^0|^{N+2s}}-\sum_{k=1}^{l-1}\sum_{i=1}^{2l-2k}\tau_i\tau_{i+2k}\frac{1}{|q_i^0-q_{i+2k}^0|^{N+2s}}.\end{aligned}$$

Notice that $\tau_i\tau_{i+2k-1} = -1$ and $\tau_i\tau_{i+2k} = 1$. Then,

$$-\sum_{i\neq j}\tau_i\tau_j\frac{1}{|q_i^0-q_j^0|^{N+2s}} = \sum_{k=1}^{l}\sum_{i=1}^{2l-(2k-1)}\frac{1}{|q_i^0-q_{i+2k-1}^0|^{N+2s}} - \sum_{k=1}^{l-1}\sum_{i=1}^{2l-2k}\frac{1}{|q_i^0-q_{i+2k}^0|^{N+2s}}.$$

Now, observe that

$$|q_i^0-q_{i+2k-1}^0| \geq \min_{i\neq j}|q_i^0-q_j^0| = \varepsilon^{-\frac{2s}{N+2s}},$$

which leads to

$$-\sum_{k=1}^{l-1}\sum_{i=1}^{2l-2k}\frac{1}{|q_i^0-q_{i+2k}^0|^{N+2s}} \geq -\varepsilon^{2s}(l-1). \tag{4.1}$$

On the other hand,

$$|q_i^0-q_{i+2k-1}^0| \leq |q_i^0-q_{i+1}^0| + \ldots + |q_{i+2k-2}^0-q_{i+2k-1}^0| \leq (2k-1)\varepsilon^{-\frac{2s}{N+2s}},$$

which gives

$$\sum_{k=1}^{l}\sum_{i=1}^{2l-(2k-1)}\frac{1}{|q_i^0-q_{i+2k-1}^0|^{N+2s}} \geq \varepsilon^{2s}\sum_{k=1}^{l}\frac{(2l-(2k-1))}{(2k-1)^{N+2s}}. \tag{4.2}$$

Gathering (4.1) and (4.2), we conclude

$$\begin{aligned}
-\sum_{i\neq j}\tau_i\tau_j\frac{1}{|q_i^0-q_j^0|^{N+2s}} &\geq \varepsilon^{2s}\left(\sum_{k=1}^{l}\frac{(2l-(2k-1))}{(2k-1)^{N+2s}} - l + 1\right)\\
&= \varepsilon^{2s}\left(l + \sum_{k=2}^{l}\frac{(2l-(2k-1))}{(2k-1)^{N+2s}}\right)\\
&> 0.
\end{aligned}$$

Hence, going back to the main estimate, taking into account the choice of $\mathbf{q}^0$ and its properties, by Lemma 4.1 and Proposition 4.2, we obtain

$$\mathcal{J}_\varepsilon(\mathbf{q}_\varepsilon) = \min_{\mathbf{q}\in\overline{\Lambda}_\varepsilon}\mathcal{J}_\varepsilon(\mathbf{q}) \leq \mathcal{J}_\varepsilon(\mathbf{q}^0) \leq 2\ell V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) + C\,\varepsilon^{\min\{2s,\frac{2N}{N+2s}\}}. \tag{4.3}$$

Now, we claim that $\mathbf{q}_\varepsilon \in \Sigma_\varepsilon$. Let us suppose, by contradiction, that $\mathbf{q}_\varepsilon \notin \Sigma_\varepsilon$. Hence, we have two possibilities: either there is an index i such that $q_{i,\varepsilon} \in \partial\omega_\varepsilon$ or $|q_{i,\varepsilon} - q_{j,\varepsilon}| = \varepsilon^{-\frac{s}{N+2s}}$, for some $j \neq i$.

In the first case, if $q_{i,\varepsilon} \in \partial\omega_\varepsilon$, assumption (V_2) implies that $V(Q_{i,\varepsilon}) > V(Q_0)+\mu_1$ for some $\mu_1 > 0$. According to Proposition 4.2, we have

$$\begin{aligned}
\sum_{i=1}^{2\ell}V(Q_{i,\varepsilon})^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) &= V(Q_{i,\varepsilon})^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) + \sum_{j\neq i}V(Q_{j,\varepsilon})^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w)\\
&> \sum_{j\neq i}V(Q_{j,\varepsilon})^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) + V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) + \mu_2\\
&> 2\ell\,V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}}I_1(w) + \mu_2
\end{aligned}$$

for some $\mu_2 > 0$, which is a contradiction with (4.3).

In the second case, we again invoke Proposition 4.2 and obtain

$$\mathcal{J}_\varepsilon(\mathbf{q}_\varepsilon) \geq 2\ell\,I_1(w)V(Q_0)^{\frac{p+1}{p-1}-\frac{N}{2s}} + C_0\varepsilon^s$$

for some $C_0 > 0$. Since $\varepsilon^s > \varepsilon^{\min\{2s,\frac{2N}{N+2s}\}}$ for all $\varepsilon > 0$ sufficiently small, we again obtain a contradiction with (4.3). Therefore, the proof is completed. □

Proof of Theorem 1.2. The conclusion follows from Lemma 2.6, Proposition 4.2, Proposition 4.3 and the minimization argument. □

Appendix A. Proofs of the results in subsection 2.4

Proof of Lemma 2.3. Let $\mathbf{q}$ be in Λ_ε fixed. There exist $0 < t_2 < t_1 < 1$ such that

$$|N_\varepsilon(\phi)| = |f(W+\phi) - f(W) - f'(W)\phi| = |f''(W + t_2\phi)\phi^2 t_1|.$$

Now, we consider two cases: when $|W| \geq \phi$ and otherwise. If $|W| \leq \phi$, we obtain

$$|N_\varepsilon(\phi)| \leq C|\phi|^p, \tag{A.1}$$

and if $|W| \geq \phi$, we obtain

$$|N_\varepsilon(\phi)| \leq C|\phi|^2, \tag{A.2}$$

since $f'(cW) \leq C$ for any fixed $c > 0$. Therefore, the estimate on the left hand side in (2.17) follows from (A.1) and (A.2).

On the other hand, for some $0 < t_1 < 1$, we obtain

$$|f'(W+\phi) - f'(W)| = p(p-1)|W + t_1\phi|^{p-2}|\phi| \leq C|\phi|^{\min\{p-1,1\}},$$

that leads to the estimate on the right hand side in (2.17). Besides, for $\psi \in \mathcal{C}^*$, we obtain $|\varrho^{-\mu}(f'(W+\phi) - f'(W))\psi| \leq |\phi|^{\min\{p-1,1\}}|\varrho^{-\mu}\psi|$. Therefore,

$$\|(f'(W+\phi) - f'(W))\psi\|_* \leq C\|\phi\|_*^{\min\{p-1,1\}}\|\psi\|_*.$$

Finally, since

$$\frac{\partial}{\partial q_{il}}N_\varepsilon(\phi) = (f'(W+\phi) - f'(W) - f''(W)\phi)\frac{\partial W}{\partial q_{il}} + (f'(W+\phi) - f'(W))\frac{\partial \phi}{\partial q_{il}},$$

we obtain (2.18). Therefore, the proof is completed. □

Proof of Lemma 2.4. First, note that $E_\varepsilon = E_{1,\varepsilon} + E_{2,\varepsilon}$, where

$$E_{1,\varepsilon} := \sum_{i=1}^{\ell} \tau_i(V(Q_i) - V(\varepsilon x))w_i \quad \text{and} \quad E_{2,\varepsilon} := f(W) - \sum_{i=1}^{\ell}\tau_i f(w_i). \tag{A.3}$$

A simple computation gives $\|E_{1,\varepsilon}\|_* \leq C\varepsilon^{\min\{N+2s-\sigma,1\}}$. Thus, we only need to estimate $\|E_{2,\varepsilon}\|_*$. For convenience, we work on $\Omega_i := \left\{x \in \mathbb{R}^N : w_i > w_j \quad \text{if } j \neq i\right\}$ for all $i = 1, \ldots, \ell$. Hence, in Ω_i, we obtain

$$\begin{aligned}
E_{2,\varepsilon} &\leq Cf'(w_i)\sum_{j\neq i}\frac{1}{|x-q_j|^{N+2s}} \\
&\leq C\frac{1}{(1+|x-q_i|)^{(N+2s)(p-1)+\sigma}}\sum_{j\neq i}\frac{1}{|q_i-q_j|^{N+2s-\sigma}} \\
&\leq C(\varrho(x))^\mu \kappa^{N+2s-\sigma},
\end{aligned}$$

where we have considered $\mu(N-2s) = \sigma$. Since $\|E_\varepsilon\|_* \leq \|E_{1,\varepsilon}\|_* + \|E_{2,\varepsilon}\|_*$, the estimate on the left hand side in (2.19) follows.

On the other hand, observe that

$$\nabla_{q_i}(V(Q_i) - V(\varepsilon x)) = \varepsilon|\nabla V(Q_i)|; \quad \left|\nabla_{q_i}\!\left(w\!\left(V(Q_i)^{\frac{1}{2s}}(x-q_i)\right)\right)\right| \leq C\frac{1}{(1+|x-q_i|)^{N+2s+1}},$$

$$|\varepsilon\nabla V(Q_i)w_i(x)| \leq C\frac{\varepsilon|\nabla V(Q_i)|}{(1+|x-q_i|)^{N+2s}}$$

and

$$|(V(Q_i) - V(\varepsilon x))\nabla_{q_i}w_i(x)| \leq C\frac{\varepsilon|\nabla V(Q_i)||x-q_i|}{(1+|x-q_i|)^{N+2s}}\left(\varepsilon + \frac{1}{1+|x-q_i|}\right).$$

Therefore, since $E_\varepsilon = E_{1,\varepsilon} + E_{2,\varepsilon}$ where $E_{1,\varepsilon}$ and $E_{2,\varepsilon}$ are given by (A.3), from (V_2), we obtain $\|\nabla_{\mathbf{q}}E_{1,\varepsilon}\|_* = O(\varepsilon)$. On the other hand, on the set Ω_i, we obtain

$$|\varrho(x)^{-\mu}(f'(W) - f'(w_i))\nabla_{q_i}w_i| \leq C\kappa^{N+2s-\sigma}.$$

In this way, $\|\nabla_{\mathbf{q}} E_{2,\varepsilon}\|_* \le C\kappa^{N+2s-\sigma}$. Since $\|\nabla_{\mathbf{q}} E_\varepsilon\|_* \le \|\nabla_{\mathbf{q}} E_{1,\varepsilon}\|_* + \|\nabla_{\mathbf{q}} E_{2,\varepsilon}\|_*$, the estimate on the right hand side in (2.19) follows. □

Proof of Proposition 2.5. Let us consider the operator $\mathfrak{F}_\varepsilon : \mathcal{A}_r \to \mathcal{C}^*$ defined by $\mathfrak{F}_\varepsilon(\phi) = T_\varepsilon(N_\varepsilon(\phi) + E_\varepsilon)$, where T_ε is given by Proposition 2.1 and

$$\mathcal{A}_r := \{\phi \in \mathcal{C}^* : \|\phi\|_* \le r\}$$

for a suitable $0 < r \ll 1$, which we will choose later. Note that if we are able to show that $\mathfrak{F}_\varepsilon$ is a contraction, then we shall obtain that there is a fixed point in $\mathcal{A}_r$ for $\mathfrak{F}_\varepsilon$, which is equivalent to solving (2.10). We have

$$\|\mathfrak{F}_\varepsilon(\phi)\|_* \le \|T_\varepsilon\left(N_\varepsilon(\phi) + E_\varepsilon\right)\|_* \le C\|N_\varepsilon(\phi) + E_\varepsilon\|_* \le C_1\left(r^{\min\{p,2\}} + \|E_\varepsilon\|_*\right).$$

Additionally, we note that

$$\|\mathfrak{F}_\varepsilon(\phi_1) - \mathfrak{F}_\varepsilon(\phi_2)\|_* \le C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_* \quad \text{for } \phi_1, \phi_2 \in \mathcal{A}_r.$$

Hence, $\mathfrak{F}_\varepsilon$ is a contraction if N_ε also it is. Observe that $|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)| = |N_\varepsilon'(\varphi)|\,|\phi_1 - \phi_2|$ for some φ on the line that joins ϕ_1 with ϕ_2. In this way, from (2.17) it follows that

$$\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_* \le C\|\varphi\|_*^{\min\{p-1,1\}}\|\phi_1 - \phi_2\|_*,$$

and then $\|\mathfrak{F}_\varepsilon(\phi_1) - \mathfrak{F}_\varepsilon(\phi_2)\|_* \le Cr^{\min\{p-1,1\}}\|\phi_1 - \phi_2\|_*$.
Now, choosing a suitable $r > 0$, for $\varepsilon > 0$ sufficiently small we obtain $\|\mathfrak{F}_\varepsilon(\phi)\|_* \le r$ for all $\phi \in \mathcal{A}_r$, and $\|\mathfrak{F}_\varepsilon(\phi_1) - \mathfrak{F}_\varepsilon(\phi_2)\|_* < \|\phi_1 - \phi_2\|_*$ for $\phi_1, \phi_2 \in \mathcal{A}_r$.

Concerning the differentiability properties, recall that ϕ is defined by the relation

$$B_\varepsilon(\phi, \mathbf{q}) := \phi - T_\varepsilon(N_\varepsilon(\phi) + E_\varepsilon) = 0.$$

Hence, $\nabla_\phi B_\varepsilon(\phi, \mathbf{q})[\eta] = \eta - T_\varepsilon(\eta\, N_\varepsilon'(\phi)) := \eta + M_\varepsilon(\eta)$, where $M_\varepsilon(\eta) = -T_\varepsilon(\eta N_\varepsilon'(\phi))$. Now, by using the fact that $\phi \in \mathcal{A}_r$, from (2.17) we get $\|M_\varepsilon(\eta)\|_* \le C\|\eta\|_*^{\min\{p-1,1\}}$. This implies that for small ε, the linear operator $\nabla_\phi B_\varepsilon(\phi, \mathbf{q})$ is invertible in $\mathcal{C}^*$, with a uniformly bounded inverse depending continuously on its parameters. Then, by applying the implicit function theorem, we obtain that $\phi(\mathbf{q})$ is a C^1-function into $\mathcal{C}^*$, with $\nabla_{\mathbf{q}}\phi = -(\nabla_\phi B_\varepsilon(\phi, \mathbf{q}))^{-1}(\nabla_{\mathbf{q}} B_\varepsilon(\phi, \mathbf{q}))$. Since

$$\nabla_{\mathbf{q}} B_\varepsilon(\phi, \mathbf{q}) = -\nabla_{\mathbf{q}} T_\varepsilon(N_\varepsilon(\phi) + E_\varepsilon) - T_\varepsilon(\nabla_{\mathbf{q}} N_\varepsilon(\phi) + \nabla_{\mathbf{q}} E_\varepsilon),$$

where all these expressions depend continuously on their parameters, it follows that

$$\|\nabla_{\mathbf{q}}\phi\|_* \le C(\|N_\varepsilon(\phi)\|_* + \|E_\varepsilon\|_* + \|\nabla_{\mathbf{q}} N_\varepsilon(\phi)\|_* + \|\nabla_{\mathbf{q}} E_\varepsilon\|_*),$$

and using the first part of this proposition, the estimates (2.17), (2.18) and (2.19), Proposition 2.1 and the constraints (2.6), we have completed the proof. □

Appendix B. Proof of Lemma 2.6

Proof of Lemma 2.6. First, we assume that u_ε is a solution of (1.1) or, equivalently, that v_ε given by (2.21) solves (2.22). Then, for each $i = 1, 2, \ldots, \ell$, it follows that

$$\nabla_{\mathbf{q}} J_\varepsilon(\widetilde{v}_\varepsilon)\left[\widetilde{\frac{\partial v_\varepsilon}{\partial q_{il}}}\right] = 0.$$

In other words,

$$\begin{aligned}
\frac{\partial \mathcal{J}_\varepsilon}{\partial q_{il}}(\mathbf{q}) &= b_s \iint_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\widetilde{v}_\varepsilon \nabla\left(\widetilde{\frac{\partial v_\varepsilon}{\partial q_{il}}}\right) dx\,dt + \int_{\mathbb{R}^N}\left(V(\varepsilon x)v_\varepsilon\frac{\partial v_\varepsilon}{\partial q_{il}} - f(v_\varepsilon)\frac{\partial v_\varepsilon}{\partial q_{il}}\right) dx \\
&= \int_{\mathbb{R}^N}\left((-\Delta)^s v_\varepsilon + V(\varepsilon x)v_\varepsilon - f(v_\varepsilon)\right)\frac{\partial v_\varepsilon}{\partial q_{il}}\, dx \\
&= 0,
\end{aligned}$$

for all i, j, which implies that $\mathbf{q}$ is a critical point of $\mathcal{J}_\varepsilon$. On the other hand, if $\mathbf{q}$ is a critical point of $\mathcal{J}_\varepsilon$, then, for v_ε given by (2.21), from (2.10) we have that

$$\nabla_{\mathbf{q}} J_\varepsilon(\widetilde{v}_\varepsilon)\left[\widetilde{\frac{\partial v_\varepsilon}{\partial q_{il}}}\right] = \sum_{j=1}^{\ell}\sum_{k=1}^{N} c_{jk} Z_{jk} \frac{\partial (W+\phi)}{\partial q_{il}} = 0 \quad \text{for all } i, l,$$

or equivalently

$$\sum_{j=1}^{\ell}\sum_{k=1}^{N} c_{jk}\left((-\Delta)^s Z_{jk}\, Z_{il} + V(\varepsilon x) Z_{jk} Z_{il} + o(1)\right) = 0 \quad \text{for all } i, l,$$

where $o(1) \to 0$ uniformly in the $\|\cdot\|_*$-norm since $\frac{\partial (W+\phi)}{\partial q_{il}} = -\tau_i Z_{il} + o(1)$. Now, noticing that the last system on the c_{il}'s is almost diagonal, we can conclude that $c_{il} = 0$ for all i, l, and therefore v_ε solves (2.22). □

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